A GLOBALLY CONVERGENT PRIMAL-DUAL INTERIOR-POINT FILTER METHOD FOR NONLINEAR PROGRAMMING: NEW FILTER OPTIMALITY MEASURES AND COMPUTATIONAL RESULTS

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Abstract. In this paper we modify the original primal-dual interior-point filter method proposed in [18] for the solution of nonlinear programming problems. We introduce two new optimality filter entries based on the objective function, and thus better suited for the purposes of minimization, and propose conditions for using inexact Hessians. We show that the global convergence properties of the method remain true under such modifications.

We also introduce a new optimization solver for the solution of nonlinear programming problems, called `ipfilter`, based on our primal-dual interior-point filter approach. The numerical results reported show that `ipfilter` is competitive both in efficiency and robustness and can handle large instances.

Key words. interior-point methods, primal-dual, filter, global convergence, large-scale NLP

1. Introduction. Interior-point methods for nonlinear programming have received recently much attention [13, 15]. A number of papers have been published studying the global convergence properties of interior-point methods for nonlinear programming [1, 2, 4, 8, 17, 20, 23]. Various codes for large-scale nonlinear programming are based on interior-point algorithms [3, 19, 20]. Filter methods, in turn, are now well understood and used for several classes of optimization problems and within different solvers [9, 10, 11, 12]. In this paper we continue the development of our primal-dual interior-point filter approach proposed in [18]. Our motivation is both theoretical and practical. We introduce new optimality filter entries better suited for minimization purposes and analyze its impact on the global convergence theory. We show how to use our approach to handle approximations to the Hessian of the Lagrangian. We are encouraged by the numerical results obtained so far for dense and sparse nonlinear programs of different types and scales.

The method in [18] belongs to the class of the so-called Newton primal-dual interior-point algorithms. It incorporates a filter technique and a line search for the purposes of globalization. It relies on a novel decomposition of the primal-dual step, obtained from the perturbed first-order necessary conditions, into a normal component and a tangential component. The normal component can be seen as a step towards the quasi-central path, i.e., the set of central strictly feasible points, whereas the tangential component aims at reducing duality, i.e., the size of the gradient of the Lagrangian, and complementarity. The line search acts on both components. All new iterates generated by the method must be acceptable to the filter and lie in a neighborhood of the quasi-central path, which is used frequently in infeasible primal-dual methods for linear and quadratic programming. Each entry in the filter is a pair of coordinates: one resulting from feasibility and centrality and associated with the normal step; the other resulting from optimality, i.e., complementarity and duality, and related to the tangential step. It has been proved that the method is globally convergent to first-order critical points. The method incorporates the possibility of entering a restoration phase, where the

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goal is the computation of a point in the central neighborhood that is acceptable to the filter and that is not too infeasible. This interior-point methodology is based on a dynamic update of the barrier parameter.

In this paper we explore the proposed framework further, introducing new optimality filter entries better tailored to the purposes of minimization. In fact, the optimality filter entry in [18] mentioned above is based on first-order principles and thus might have weaknesses in distinguishing well between minimization and maximization. The new optimality filter entries are obtained by adding to the objective function or the Lagrangian function a constant multiple of complementarity. These optimality filter entries are closer to the ones used in SLP/SQP filter methods where the optimality filter entry is given by the objective function. We show in this paper that the method retains its global convergence properties when using the new optimality filter entries, under a uniform condition on the positive semi-definiteness of the Hessian of the Lagrangian on the appropriate null space of the constraints.

Another aspect in [18] that we improve here is the dependence of the algorithm and convergence theory on the use of exact Hessians of the Lagrangian. We study how the framework in [18] can be adapted to the absence of second-order derivatives. The only component of the algorithm, when using the new optimality filter entries, which becomes critical without using exact Hessians of the Lagrangian is the need to keep the iterates in the central neighborhood. We have thus explored this fact to our advantage and formulate practical inexact conditions, to use when second-order derivatives are unavailable, based on the action of the inexact Hessian along an appropriate vector.

Finally, in this paper we also report numerical results of `ipfilter`, the Fortran 90 implementation of our primal-dual interior-point filter approach. We tested `ipfilter` on a set of constrained nonlinear programs from CUTEr [14], including large instances. The results are compared against those obtained by `ipopt`, the barrier interior-point filter code developed by Wächter and Biegler [21]. The results show that the current initial version of `ipfilter` is already competitive in terms of robustness and number of primal-dual iterations. The `ipfilter` solver is freely available for academic and research purposes and has been selected as one of the solvers of a recent European Space Agency project which aims to produce a general purpose robust NLP solver especially tailored for space trajectory optimization. The `ipfilter` web site is located at:

$$\texttt{http://www.mat.uc.pt/ipfilter} \tag{1.1}$$

The paper begins with a description of the interior-point filter framework in Section 2 where we also introduce the new optimality filter entries and address the use of second-order derivatives. We prove in Section 2, under appropriate assumptions, that the primal-dual step is a descent direction for these new filter entries for points in the central neighborhood which are not too infeasible. We also prove in this section that the adjustments in the definition of the central neighborhood required to handle approximations to second-order derivatives still allow large enough step lengths. The (modified) method is then described in detail in Section 3. The analysis of global convergence for the primal-dual interior-point method, modified to handle the new optimality filter entries and the relaxation of the Hessian requirement, is given in Section 4. The last two sections of the paper concern the development and testing of the `ipfilter` solver.

2. Interior-point framework. For the purpose of describing our algorithm and deriving the corresponding analysis of global convergence, we write a general nonlinear programming problem in the form

$$\min f(x) \quad \text{s.t.} \quad h(x) = 0, \quad x \geq 0, \tag{2.1}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are twice continuously differentiable functions on an open set $\Omega \subset \mathbb{R}^n$. The implementation of our algorithm in the `ipfilter` code converts first any nonlinear programming problem where the feasible region involves inequalities not of the simple bound type in a problem with equalities and simple bounds (of the form $l \leq x \leq u$) by means of slack variables. For simplicity, in the presentation of the algorithm and of the analysis of global convergence we deal only with simple bounds of the form $x \geq 0$.

2.1. Step computation. Primal-dual interior-point methods are derived by applying Newton's method to an appropriate perturbation of the first-order Karush-Kuhn-Tucker or KKT conditions (which under appropriate constraint qualifications are known to be necessary for local minimizers). Let us write the KKT conditions of problem (2.1) in the form

$$\nabla_x \ell(x, y, z) = 0, \tag{2.2}$$

$$h(x) = 0, \tag{2.3}$$

$$Xz = 0, \tag{2.4}$$

$$x \geq 0, \quad z \geq 0, \tag{2.5}$$

where $X = \mathrm{diag}(x)$, $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$ are the Lagrange multipliers, and ℓ denotes the Lagrangian function

$$\ell(x, y, z) = f(x) + h(x)^T y - x^T z.$$

The above mentioned perturbation is made in the complementarity conditions (block (2.4) of the KKT system (2.2)-(2.4)):

$$F_{\sigma\mu}(x, y, z) \stackrel{\mathrm{def}}{=} \begin{pmatrix} \nabla_x \ell(x, y, z) \\ h(x) \\ Xz - \sigma\mu e \end{pmatrix} = 0,$$

where, as in [18], $\sigma \in (0, 1)$ plays the role of a centering parameter and μ is a measure of complementarity

$$\mu = \frac{x^T z}{n}. \tag{2.6}$$

We also use the notation

$$w = (x, y, z) \quad \text{and} \quad \Delta w = (\Delta x, \Delta y, \Delta z).$$

In this paper the primal-dual step Δw is computed by solving an approximated linearized perturbed KKT system, of the form

$$\begin{pmatrix} H & \nabla h(x) & -I \\ \nabla h(x)^T & 0 & 0 \\ Z & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = - \begin{pmatrix} \nabla_x \ell(x, y, z) \\ h(x) \\ Xz - \sigma\mu e \end{pmatrix},$$

where H is an approximation to $\nabla^2_{xx} \ell(x, y, z)$ which will be required to satisfy certain conditions. We denote the matrix of this system by $\mathrm{KKT}(w)$. When $H = \nabla^2_{xx} \ell(w)$, we have that $\mathrm{KKT}(w) = F'_{\sigma\mu}(w)$. As discussed in [18], the choice of complementarity measure μ according to (2.6) ensures that the primal-dual step Δw is a descent direction for $x^T z/n$, allowing a dynamic reduction of μ (see also [8]).

The approach introduced in [18] to adapt the methodology of a filter to the interior-point context specified two quantities for the filter entries, the first component corresponding to quasi-centrality (feasibility and centrality) and the second corresponding to optimality (complementarity and criticality). This choice of filter components was then associated with a decomposition of the trial step into a normal step and a tangential step that yielded a decrease on the respective filter components. In fact, the perturbed KKT-conditions can be rewritten as

$$F_{\sigma\mu}(x,y,z) = \begin{pmatrix} 0 \\ h(x) \\ Xz - \mu e \end{pmatrix} + \begin{pmatrix} \nabla_x \ell(x,y,z) \\ 0 \\ (1-\sigma)\mu e \end{pmatrix} = 0. \tag{2.7}$$

This splitting motivated the step decomposition $\Delta w = s^n + s^t$, where the normal step $s^n = (\Delta x^n, \Delta y^n, \Delta z^n)$ is the solution of

$$\mathrm{KKT}(w) s^n = - \begin{pmatrix} 0 \\ h(x) \\ Xz - \mu e \end{pmatrix}, \tag{2.8}$$

and the tangential step $s^t = (\Delta x^t, \Delta y^t, \Delta z^t)$ is given by

$$\mathrm{KKT}(w) s^t = - \begin{pmatrix} \nabla_x \ell(w) \\ 0 \\ (1-\sigma)\mu e \end{pmatrix}. \tag{2.9}$$

2.2. New filter entries. The first term in the middle expression of (2.7) measures the proximity to the quasi-central path [8] and led to the choice of the following filter component

$$\theta(w) = \|h(x)\| + \|Xz - (x^T z)/ne\|.$$

The second term in the middle expression of (2.7) measures complementarity and criticality. Thus, we chose in [18], for the second filter component, the optimality measure

$$x^T z/n + \|\nabla_x \ell(w)\|^2. \tag{2.10}$$

The choice (2.10) arose naturally given the decomposition of the step into its normal and tangential components. However, since it is based on first-order principles, it might not distinguish sufficiently well between local minimization and local maximization. One alternative we analyze in this paper is given by

$$\theta_g(w) = f(x) + h(x)^T y + c\mu = \ell(x,y,z) + (c+n)\mu, \tag{2.11}$$

where $c > 0$ is a given constant to be specified later. Another alternative is simply

$$\theta_g(w) = f(x) + c\mu. \tag{2.12}$$

These alternatives are closer to the original choice of $\theta_g(w) = f(x)$ used in SLP/SQP filter methods. In fact, note that when μ is small in (2.12) (or when μ and $\|h(x)\|$ are small in (2.11) and the size of y is moderate), then $\theta_g(w) \simeq f(x)$.

2.3. Step length. The flexibility of the step splitting was used in [18] to introduce different step sizes for s^n and s^t in the trial step computation. Let Δ be the positive scalar that controls the length of the step taken along Δw, forcing the damped components $\alpha^n(\Delta)s^n$ and $\alpha^t(\Delta)s^t$, to satisfy

$$\|\alpha^n(\Delta)s^n\| \leq \Delta, \qquad \|\alpha^t(\Delta)s^t\| \leq \Delta.$$

Having these bounds in mind, and requiring explicitly $\alpha^t(\Delta) \leq \alpha^n(\Delta)$, the step sizes taken along the normal and tangential directions respectively are defined as

$$\alpha^n(\Delta) = \min\left\{1, \frac{\Delta}{\|s^n\|}\right\}, \tag{2.13}$$

$$\alpha^t(\Delta) = \min\left\{\alpha^n(\Delta), \frac{\Delta}{\|s^t\|}\right\} = \min\left\{1, \frac{\Delta}{\|s^n\|}, \frac{\Delta}{\|s^t\|}\right\}. \tag{2.14}$$

Here, we use for $\Delta > 0$ the natural definition $\alpha^n(\Delta) = 1$ for $\|s^n\| = 0$, by using the convention $\min\{1, \infty\} = 1$. We also say that $\alpha^t(\Delta) = \alpha^n(\Delta)$ if $\|s^t\| = 0$, although our algorithm cannot generate tangential steps for which $\|s^t\| = 0$ since the right-hand-side in (2.9) will never be zero if the iterates x and z are kept positive throughout. The requirement $\alpha^t(\Delta) \leq \alpha^n(\Delta)$ is mainly necessary to enforce the iterates to stay in the neighborhood $\mathcal{N}(\gamma, M, p)$ defined in (2.18) below, see Lemma 2.7.

Let also

$$w(\Delta) = (x(\Delta), y(\Delta), z(\Delta)) = w + \alpha^n(\Delta)s^n + \alpha^t(\Delta)s^t, \tag{2.15}$$

$$s(\Delta) = \big(s_x(\Delta), s_y(\Delta), s_z(\Delta)\big) = w(\Delta) - w = \alpha^n(\Delta)s^n + \alpha^t(\Delta)s^t. \tag{2.16}$$

Thus, $\|s(\Delta)\| \leq 2\Delta$ (and one can see that Δ plays a role comparable to a trust-region radius).

The scalars $\alpha^n(\Delta)$ and $\alpha^t(\Delta)$ will be such that positivity and some measure of centrality of the new iterate $w(\Delta)$ are maintained. However, both $\alpha^n(\Delta)$ and $\alpha^t(\Delta)$ depend on Δ, that in turn will be adjusted, not only to meet the purpose of positivity and centrality, but also to enforce global convergence.

We introduce the notation

$$\theta_h(w) = \|h(x)\|, \qquad \theta_c(w) = \left\|Xz - \frac{x^T z}{n}e\right\|, \qquad \theta_\ell(w) = \|\nabla_x \ell(w)\|,$$

which allows us to write the first filter component as

$$\theta(w) = \theta_c(w) + \theta_h(w),$$

Note that a point w satisfying $\theta(w) = \theta_\ell(w) = 0$, $\mu = 0$, and $(x, z) \geq 0$, is a KKT point.

With the purpose of achieving a reduction on the function θ_g, we introduce, at a given point w, the linear model

$$m(w(\Delta)) = \theta_g(w) + \nabla\theta_g(w)^T(w(\Delta) - w).$$

To simplify the notation we also define

$$\mu(\Delta) = \frac{x(\Delta)^T z(\Delta)}{n}.$$

2.4. Central neighborhood. One possibility to prevent $(x(\Delta), z(\Delta))$ from approaching the boundary of the positive orthant too rapidly is to keep the iterates in some form of central neighborhood. In [18], we have used the neighborhood

$$\mathcal{N}(\gamma, M) = \left\{w\ : (x, z) > 0,\ \ Xz \geq \gamma\frac{x^T z}{n},\ \ \theta_h(w) + \theta_\ell(w) \leq M\frac{x^T z}{n}\right\}, \tag{2.17}$$

with fixed $\gamma \in (0, 1)$ and $M > 0$ (see also the references [8, 22]). To keep the iterates in this neighborhood we need the primal-dual step to yield some form of sufficient decrease on θ_ℓ. In particular, we were able to prove in [18] for steps using exact second-order derivatives that

$$\theta_\ell(w(\Delta)) \leq (1 - \alpha^t(\Delta))\theta_\ell(w) + M_\ell\Delta^2,$$

for some constant M_ℓ depending on the Lipschitz constants of the second-order derivatives of the functions defining the problem.

In this paper we will consider a more general scenario which will allow us to work with different types of approximations H to the Hessian of the Lagrangian. For this purpose, we consider a family of neighborhoods parameterized by $p \in [1,2]$:

$$\mathcal{N}(\gamma, M, p) = \left\{ w \,:\, (x,z) > 0,\;\; Xz \geq \gamma \frac{x^T z}{n},\;\; \theta_h(w) + \theta_\ell(w)^p \leq M \frac{x^T z}{n} \right\}. \qquad (2.18)$$

When $p = 1$ we recover (2.17), i.e., we have that $\mathcal{N}(\gamma, M, 1) = \mathcal{N}(\gamma, M)$.

We then ask H to satisfy a sufficient decrease condition on θ_ℓ^p of the following type

$$\theta_\ell(w(\Delta))^p \leq (1 - p\alpha^t(\Delta))\theta_\ell(w)^p + M_\ell \max\{\Delta^q, \Delta^2\}, \qquad (2.19)$$

where $q \in (1,2]$. When $p = 1$ and $q = 2$, we recover the case treated in [18].

Similarly to what has been showed in [18] for the neighborhood (2.17), we will see in the next subsection that $w \in \mathcal{N}(\gamma, M, p)$ implies $w(\Delta) \in \mathcal{N}(\gamma, M, p)$ whenever $\Delta > 0$ is sufficiently small.

2.5. Use of second-order derivatives. It was proved in [18, Lemma 1] that (2.19) is true for all $\Delta > 0$ when $p = 1$, $q = 2$, and $H = \nabla^2_{xx}\ell(w)$. The constant M_ℓ in (2.19) depends on the Lipschitz constant of the Hessian of the Lagrangian. It is easy to see that (2.19) is also true, when $p = 1$, for all values of $\Delta > 0$ such that

$$\|H - \nabla^2_{xx}\ell(w)\| \leq N\Delta^{q-1}, \qquad (2.20)$$

where N is any fixed positive constant and $q \in (1,2]$. In this case, M_ℓ also depends on N.

Moreover, (2.19) is also true, when $p = 2$, for all values of $\Delta > 0$ such that

$$\|[H - \nabla^2_{xx}\ell(w)]\nabla_x\ell(w)\| \leq N\|\nabla_x\ell(w)\|\Delta^{q-1}, \qquad (2.21)$$

where N is any fixed positive constant and $q \in (1,2]$. More generally, we have the following result.

LEMMA 2.1. *Let $p, q \in [1,2]$ and let H satisfy (2.21). Then the following results hold.*

i) *There exists a constant $M_\ell > 0$, that only depends on N, on bounds for θ_ℓ and $\|\nabla^2_{xw}\ell\|$, and on a Lipschitz constant for $\nabla^2_{xw}\ell$, such that*

$$\theta_\ell^p(w(\Delta)) \leq (1 - p\alpha^t(\Delta))\theta_\ell^p(w) + M_\ell \max\{\Delta^p, \Delta^q, \Delta^2\}. \qquad (2.22)$$

ii) *If in addition $\theta_\ell(w) \geq \epsilon > 0$ then there exists a constant $M'_{\ell,p} > 0$ that only depends on p and ϵ, on N, on bounds for θ_ℓ and $\|\nabla^2_{xw}\ell\|$, and on a Lipschitz constant for $\nabla^2_{xw}\ell$, such that*

$$\theta_\ell^p(w(\Delta)) \leq (1 - p\alpha^t(\Delta))\theta_\ell^p(w) + M'_{\ell,p} \max\{\Delta^q, \Delta^2\}. \qquad (2.23)$$

For the proof we need the following auxiliary result.

LEMMA 2.2. *Let $p \in [1,2]$ and $g(x) = \|x\|^p$. Then g is infinitely differentiable on $\mathbb{R}^n \setminus \{0\}$ with*

$$\nabla g(x) = p\|x\|^{p-2}x, \quad \nabla^2 g(x) = p\|x\|^{p-2}I + p(p-2)\|x\|^{p-4}xx^T. \qquad (2.24)$$

If $p > 1$ then g is also continuously differentiable at $x = 0$ with $\nabla g(0) = 0$, and if $p = 2$, g is infinitely differentiable at $x = 0$ with $\nabla^2 g(0) = 2I$.

Now let x, y be given ($x, y \neq 0$ if $p = 1$). Then

$$\|\nabla g(y) - \nabla g(x)\| \leq 2^{3-p} p \|y - x\|^{p-1}. \tag{2.25}$$

Furthermore, if $\rho \stackrel{\text{def}}{=} \min_{0 \leq t \leq 1} \|(1-t)x + ty\| > 0$, there also holds

$$\|\nabla g(y) - \nabla g(x)\| \leq p \rho^{p-2} \|y - x\|. \tag{2.26}$$

The proof of this auxiliary lemma is given in the appendix.

Proof. (of Lemma 2.1) Let B_ℓ and $B_{\ell'}$ be bounds for θ_ℓ and $\|\nabla^2_{xw}\ell\|$, respectively. Then $B_{\ell'}$ is also a Lipschitz constant for $\nabla_x \ell$. Denote by $C_{\ell'}$ a Lipschitz constant for $\nabla^2_{xw}\ell$.

We prove the result first for $p > 1$. Limit transition $p \downarrow 1$ yields then the case $p = 1$. We will apply Lemma 2.2 with $g(x) = \|x\|^p$.

We first prove the estimate

$$\begin{aligned} &\nabla g(\nabla_x \ell(w))^T \nabla^2_{xw} \ell(w) s(\Delta) \leq -p\alpha^t(\Delta) \theta^p_\ell(w) + R_1(\Delta) \\ &\text{with} \quad R_1(\Delta) = 2pN\theta^{p-1}_\ell(w) \Delta^q \leq 2pNB^{p-1}_\ell \Delta^q. \end{aligned} \tag{2.27}$$

Consider first the case $\nabla_x \ell(w) = 0$. Then (2.27) holds true, since the left-hand side is zero and the right-hand side reduces to $R_1(\Delta)$, which is zero, too.

In the case $\nabla_x \ell(w) \neq 0$ there holds

$$\begin{aligned} \nabla g(\nabla_x \ell(w))^T \nabla^2_{xw} \ell(w) s(\Delta) &= p \|\nabla_x \ell(w)\|^{p-2} \nabla_x \ell(w)^T \nabla^2_{xw} \ell(w) s(\Delta) \\ &\leq p \|\nabla_x \ell(w)\|^{p-2} \nabla_x \ell(w)^T (-\alpha^t(\Delta) \nabla_x \ell(w)) + pN \|\nabla_x \ell(w)\|^{p-1} \Delta^{q-1} \|s(\Delta)\|. \end{aligned}$$

From this (2.27) follows immediately.

Next, we use (2.27) to derive the estimate

$$\begin{aligned} \theta^p_\ell(w(\Delta)) &= \theta^p_\ell(w) + \int_0^1 \nabla g(\nabla_x \ell(w + ts(\Delta)))^T \nabla^2_{xw} \ell(w + ts(\Delta)) s(\Delta)\, dt \\ &= \theta^p_\ell(w) + \nabla g(\nabla_x \ell(w))^T \nabla^2_{xw} \ell(w) s(\Delta) + R_2(\Delta) \\ &\leq (1 - p\alpha^t(\Delta)) \theta^p_\ell(w) + R_1(\Delta) + R_2(\Delta), \end{aligned}$$

where

$$\begin{aligned} |R_2(\Delta)| \leq &\int_0^1 |[\nabla g(\nabla_x \ell(w + ts(\Delta)))^T \nabla^2_{xw} \ell(w + ts(\Delta)) \\ &\qquad - \nabla g(\nabla_x \ell(w))^T \nabla^2_{xw} \ell(w)] s(\Delta)|\, dt \\ \leq &\int_0^1 |[\nabla g(\nabla_x \ell(w + ts(\Delta))) - \nabla g(\nabla_x \ell(w))]^T \nabla^2_{xw} \ell(w + ts(\Delta)) s(\Delta)|\, dt \\ + &\int_0^1 |\nabla g(\nabla_x \ell(w)))^T (\nabla^2_{xw} \ell(w + ts(\Delta)) - \nabla^2_{xw} \ell(w)) s(\Delta)|\, dt \\ \stackrel{\text{def}}{=} &\; R_3(\Delta) + R_4(\Delta). \end{aligned}$$

Now with (2.25) and (2.24)

$$\begin{aligned} R_3(\Delta) &\leq \frac{1}{p} B_{\ell'} 2^{3-p} p B^{p-1}_{\ell'} \|s(\Delta)\|^{p-1} \|s(\Delta)\| \leq 8 B^p_{\ell'} \Delta^p, \\ R_4(\Delta) &\leq \frac{1}{2} p \|\nabla_x \ell(w)\|^{p-1} C_{\ell'} \|s(\Delta)\|^2 \leq 2pB^{p-1}_\ell C_{\ell'} \Delta^2. \end{aligned}$$

This shows the first assertion.

2. To prove the second assertion we estimate, for all $t \in [0,1]$,

$$\|\nabla_x \ell(w + ts(\Delta)) - \nabla_x \ell(w)\| \le \int_0^t \|\nabla_{xw}^2 \ell(w + \tau s(\Delta)) s(\Delta)\| \, d\tau \le 2tB_{\ell'}\Delta.$$

Hence, if we choose $\Delta \le \Delta_\ell \stackrel{\text{def}}{=} \epsilon/(4B_{\ell'})$, we have

$$\|\nabla_x \ell(w + ts(\Delta)) - \nabla_x \ell(w)\| \le \epsilon/2 \quad \forall\, t \in [0,1].$$

Therefore, for all $\tau, t \in [0,1]$,

$$\begin{aligned}&\|(1-\tau)\nabla_x \ell(w) + \tau \nabla_x \ell(w + ts(\Delta))\| \\ &\ge \|\nabla_x \ell(w)\| - \tau \|\nabla_x \ell(w + ts(\Delta)) - \nabla_x \ell(w)\| \ge \epsilon - \tau\epsilon/2 \ge \epsilon/2.\end{aligned}$$

By using (2.26), we then obtain, for all $\Delta \le \Delta_\ell$,

$$R_3(\Delta) \le \frac{1}{2} B_{\ell'} p \left(\frac{\epsilon}{2}\right)^{p-2} B_{\ell'} \|s(\Delta)\|^2 \le 2B_{\ell'}^2 p \left(\frac{\epsilon}{2}\right)^{p-2} \Delta^2.$$

This concludes the proof of the second assertion. □

Let us now discuss the context of condition (2.21). The use of exact Hessians $H = \nabla_{xx}^2 \ell(w)$ or of Hessian approximations H such that (2.20) holds are certainly two ways to satisfy (2.21). However, this condition can be imposed without requiring the Hessian of the Lagrangian or an entire approximation thereof. In fact, we can first calculate the matrix-vector product

$$r(w) = \nabla_{xx}^2 \ell(w) \nabla_x \ell(w)$$

using finite difference approximations with an error of the order of Δ^{q-1}:

$$\|\tilde{r}(w) - r(w)\| = \mathcal{O}(\Delta^{q-1}).$$

Then, we require H to satisfy

$$H\tilde{r}(w) = \nabla_x \ell(w), \tag{2.28}$$

and there are several ways to achieve this last goal. For instance, one can compute H from a quasi-Newton update where this condition is additionally imposed. The numerical experiments reported in this paper were performed on the CUTEr collection test where second-order derivatives are available. A numerical study on the imposition of (2.21) without second-order derivatives is out of the scope of this paper. Condition (2.21) seems to be the strongest relaxation of our algorithmic framework to the case where only first-order derivatives are available for which one can prove global convergence to first-order stationary points.

2.6. Step estimates. We start by recalling the result in [18, Lemma 13] which estimates the variation on the complementarity measure μ along the primal-dual step.

LEMMA 2.3. *For all $\Delta > 0$ it holds*

$$\begin{aligned}&X(\Delta)z(\Delta) \le \big(\gamma + (1-\gamma)\alpha^n(\Delta) - \alpha^t(\Delta)(1-\sigma)\big)\mu e + 4\Delta^2 e,\\ &X(\Delta)z(\Delta) \ge \big(\gamma + (1-\gamma)\alpha^n(\Delta) - \alpha^t(\Delta)(1-\sigma)\big)\mu e - 4\Delta^2 e,\\ &\mu(\Delta) \le \big(1 - \alpha^t(\Delta)(1-\sigma)\big)\mu + 4\Delta^2,\\ &\mu(\Delta) \ge \big(1 - \alpha^t(\Delta)(1-\sigma)\big)\mu - 4\Delta^2.\end{aligned} \tag{2.29}$$

The following lemma estimates the variation in θ_h and θ_c along the primal-dual step — the proofs are exactly as the corresponding ones in [18, Lemma 1].

LEMMA 2.4. *There exist positive constants M_h (depending on the Lipschitz constant of ∇h) and M_c such that, for all $\Delta > 0$,*

$$\theta_h(w(\Delta)) \leq (1-\alpha^n(\Delta))\theta_h(w) + M_h\Delta^2, \tag{2.30}$$

$$\theta_c(w(\Delta)) \leq (1-\alpha^n(\Delta))\theta_c(w) + M_c\Delta^2. \tag{2.31}$$

The next lemma is crucial in the analysis of global convergence of our primal-dual interior-point filter method when using the new optimality filter entries (2.11) and (2.12) since it analysis the behavior of these quantities along the primal-dual step. It gives an upper bound for these two filter entries θ_g at the new point $w(\Delta)$, in terms of Δ and of the corresponding values at the previous point w. It also provides a lower bound for the decrease produced on the linear model m by the step $w(\Delta) - w$.

LEMMA 2.5. *Let $\mathrm{KKT}(w)$ be invertible and assume that $Xz \geq \gamma\mu e$. Let also $H + \frac{1}{2}X^{-1}Z$ be positive semidefinite on the null space of $\nabla h(x)^T$. Then, for $\Delta > 0$, it holds*

$$\theta_g(w(\Delta)) - \theta_g(w) \leq -M_\mu\alpha^t(\Delta)\mu + M_\theta\theta(w) + M_g\Delta^2, \tag{2.32}$$

for some positive constants M_μ, M_θ, and M_g and for all

$$c \geq \frac{3n^2}{1-\sigma}\left(\max\left\{1, \frac{1-\sigma}{\gamma}\right\}\right)^2. \tag{2.33}$$

For any $\Delta > 0$, we also have

$$m(w) - m(w(\Delta)) \geq M_\mu\alpha^t(\Delta)\mu - M_\theta\theta(w). \tag{2.34}$$

The constant M_θ depends on upper bounds for $\|\mathrm{KKT}(w)^{-1}\|$ and $\|H\|$.

Proof. We prove the result for the measure $\theta_g(w)$ given by (2.11). The proof for (2.12) is essentially the same and differs only on the contribution of the terms to $M_\theta\theta(w)$.

To prove (2.32) we start by applying a Taylor expansion

$$\theta_g(w) - \theta_g(w(\Delta)) = -\nabla\theta_g(w)^T(w(\Delta)-w) - \mathcal{O}(\Delta^2). \tag{2.35}$$

Using the first block of equations of the system (2.9) and summing and subtracting an appropriate term at the end, we have

$$\begin{aligned}
&-\nabla\theta_g(w)^T(w(\Delta)-w)\\
&\qquad = -s_x(\Delta)^T\nabla_x\ell(w) - s_y(\Delta)^T\nabla_y\ell(w) - s_z(\Delta)^T\nabla_z\ell(w)\\
&\qquad\quad - \frac{c+n}{n}(s_x(\Delta)^Tz + s_z(\Delta)^Tx)\\
&\qquad = s_x(\Delta)^TH\Delta x^t + s_x(\Delta)^T\nabla h(x)\Delta y^t - s_x(\Delta)^T\Delta z^t\\
&\qquad\quad - s_y(\Delta)^Th(x) + s_z(\Delta)^Tx - \frac{c+n}{n}(s_x(\Delta)^Tz + s_z(\Delta)^Tx)\\
&\qquad = s_x(\Delta)^TH\Delta x^t + s_x(\Delta)^T\nabla h(x)\Delta y^t - s_x(\Delta)^T\Delta z^t\\
&\qquad\quad - s_x(\Delta)^Tz - \frac{c}{n}s_x(\Delta)^Tz - s_y(\Delta)^Th(x) - \frac{c}{n}s_z(\Delta)^Tx\\
&\qquad\quad + \frac{1}{2}\alpha^t(\Delta)(\Delta x^t)^T(X^{-1}Z)\Delta x^t - \frac{1}{2}\alpha^t(\Delta)(\Delta x^t)^T(X^{-1}Z)\Delta x^t.
\end{aligned} \tag{2.36}$$

We can decompose the term involving the Hessian of the Lagrangian in (2.36) using (2.16) as follows

$$s_x(\Delta)^T H \Delta x^t = \alpha^n(\Delta)(\Delta x^n)^T H \Delta x^t + \alpha^t(\Delta)(\Delta x^t)^T H \Delta x^t.$$

From the definition of the normal component of the step, we have

$$\alpha^n(\Delta)(\Delta x^n)^T H \Delta x^t \geq -\|\Delta x^t\| \, \|H\| \, \|\Delta x^n\| \geq -C_1 \theta(w), \tag{2.37}$$

where $C_1 > 0$ is an upper bound for $\| \mathrm{KKT}(w)^{-1} \| \|H\| \|\Delta x^t\|$. Thus, from the assumption of the lemma,

$$s_x(\Delta)^T H \Delta x^t + \frac{\alpha^t(\Delta)}{2}(\Delta x^t)^T (X^{-1} Z) \Delta x^t \geq -C_1 \theta(w). \tag{2.38}$$

Using the second block equations in (2.8) and (2.9) we have, for the second and sixth term of the last expression in (2.36) that

$$s_x(\Delta)^T \nabla h(x) \Delta y^t = -\alpha^n(\Delta) h(x)^T \Delta y^t \geq -C_2 \|h(x)\| \geq -C_2 \theta(w), \tag{2.39}$$

$$-s_y(\Delta)^T h(x) \geq -C_3 \|h(x)\| \geq -C_3 \theta(w), \tag{2.40}$$

with C_2 and C_3 positive constants representing upper bounds for the norms of Δy^t and $s_y(\Delta)$, respectively.

By the third block of equations in (2.8) and (2.9) we have

$$\begin{aligned}
-\frac{c}{n} s_x(\Delta)^T z - \frac{c}{n} s_z(\Delta)^T x &= \\
&= -\frac{c}{n}\alpha^n(\Delta)(\Delta x^n)^T z - \frac{c}{n}\alpha^t(\Delta)(\Delta x^t)^T z \\
&\quad - \frac{c}{n}\alpha^n(\Delta)(\Delta z^n)^T x - \frac{c}{n}\alpha^t(\Delta)(\Delta z^t)^T x \\
&= -\frac{c}{n}\alpha^n(\Delta)\big((\Delta x^n)^T z + (\Delta z^n)^T x\big) - \frac{c}{n}\alpha^t(\Delta)\big((\Delta x^t)^T z + (\Delta z^t)^T x\big) \\
&= -\frac{c}{n}\alpha^n(\Delta)\big((\Delta x^n)^T z + (\Delta z^n)^T x\big) - \frac{c}{n}\alpha^t(\Delta)(-n(1-\sigma)\mu) \\
&\geq -C_4 \theta(w) + [c\,(1-\sigma)]\alpha^t(\Delta)\mu,
\end{aligned} \tag{2.41}$$

where $C_4 > 0$ depends on c and on upper bounds for the norms of $\mathrm{KKT}(w)^{-1}$, x, and z.

Using again the third block of equations in (2.9), we obtain

$$
\begin{aligned}
&- s_x(\Delta)^T(\Delta z^t + z) - \frac{1}{2}\alpha^t(\Delta)(\Delta x^t)^T(X^{-1}Z)\Delta x^t = \\
&\quad = -(\alpha^n(\Delta)\Delta x^n + \alpha^t(\Delta)\Delta x^t)^T(\Delta z^t + z) - \frac{1}{2}\alpha^t(\Delta)(\Delta x^t)^T(X^{-1}Z)\Delta x^t \\
&\quad = -\alpha^n(\Delta)(\Delta x^n)^T(\Delta z^t + z) - \alpha^t(\Delta)(\Delta x^t)^T(\Delta z^t + z) \\
&\qquad - \frac{1}{2}\alpha^t(\Delta)(\Delta x^t)^T(X^{-1}Z)\Delta x^t \\
&\quad \geq -C_5\theta(w) - \alpha^t(\Delta)(\Delta x^t)^T(-X^{-1}Z\Delta x^t - (1-\sigma)X^{-1}\mu e + z) \\
&\qquad - \frac{1}{2}\alpha^t(\Delta)(\Delta x^t)^T(X^{-1}Z)\Delta x^t \\
&\quad = -C_5\theta(w) + \frac{1}{2}\alpha^t(\Delta)(\Delta x^t)^T X^{-1}Z\Delta x^t - \alpha^t(\Delta)(\Delta x^t)^T(z - (1-\sigma)X^{-1}\mu e) \\
&\quad = -C_5\theta(w) + \frac{1}{2}\alpha^t(\Delta)(Z\Delta x^t)^T(X^{-1}Z^{-1})(Z\Delta x^t) \\
&\qquad - \alpha^t(\Delta)(Z\Delta x^t)^T(e - X^{-1}Z^{-1}(1-\sigma)\mu e) \\
&\quad \geq -C_5\theta(w) + \frac{1}{2}\alpha^t(\Delta)\frac{1}{\|XZ\|}\|Z\Delta x^t\|^2 \\
&\qquad - \alpha^t(\Delta)\|Z\Delta x^t\|\|e - X^{-1}Z^{-1}(1-\sigma)\mu e\| \\
&\quad \geq -C_5\theta(w) + \alpha^t(\Delta)\frac{1}{2n\mu}\|Z\Delta x^t\|^2 - \alpha^t(\Delta)\|Z\Delta x^t\|\|e - X^{-1}Z^{-1}(1-\sigma)\mu e\| \\
&\quad = -C_5\theta(w) + \frac{1}{\mu}\alpha^t(\Delta)\|Z\Delta x^t\|\left(\frac{1}{2n}\|Z\Delta x^t\| - \mu\|e - X^{-1}Z^{-1}(1-\sigma)\mu e\|\right),
\end{aligned}
\tag{2.42}
$$

with $C_5 > 0$ an upper bound for $\|\,\mathrm{KKT}(w)^{-1}\|\|z + \Delta z^t\|$.

Now, we consider two cases.

Case 1:

$$\frac{1}{2n}\|Z\Delta x^t\| \geq \mu\|e - X^{-1}Z^{-1}(1-\sigma)\mu e\|.$$

Then, from (2.42), we have

$$-s_x(\Delta)^T(\Delta z^t + z) - \frac{1}{2}\alpha^t(\Delta)(\Delta x^t)^T(X^{-1}Z)\Delta x^t \geq -C_5\theta(w). \tag{2.43}$$

Case 2:

$$\frac{1}{2n}\|Z\Delta x^t\| < \mu\|e - X^{-1}Z^{-1}(1-\sigma)\mu e\|. \tag{2.44}$$

Then, from $Xz \geq \gamma\mu e$, we get

$$\frac{1}{x_i z_i} \leq \frac{1}{\gamma\mu}, \quad i = 1, \ldots, n,$$

and, using (2.44),

$$
\begin{aligned}
&-\alpha^t(\Delta)\|Z\Delta x^t\|\|e - X^{-1}Z^{-1}(1-\sigma)\mu e\| \\
&\qquad \geq -\alpha^t(\Delta)2n\mu\|e - X^{-1}Z^{-1}(1-\sigma)\mu e\|^2 \\
&\qquad \geq -\alpha^t(\Delta)2n\mu\left(\sqrt{n}\max\left\{1, \frac{1-\sigma}{\gamma}\right\}\right)^2 \\
&\qquad = -\alpha^t(\Delta)2n^2\mu\left(\max\left\{1, \frac{1-\sigma}{\gamma}\right\}\right)^2.
\end{aligned}
$$

Thus, from (2.42),

$$
\begin{aligned}
&-s_x(\Delta)^T(\Delta z^t + z) - \frac{1}{2}\alpha^t(\Delta)(\Delta x^t)^T(X^{-1}Z)\Delta x^t \qquad (2.45)\\
&\geq -C_5\theta(w) - \alpha^t(\Delta)\|Z\Delta x^t\|\|e - X^{-1}Z^{-1}(1-\sigma)\mu e\| \\
&\geq -C_5\theta(w) - 2n^2\left(\max\left\{1, \frac{1-\sigma}{\gamma}\right\}\right)^2\alpha^t(\Delta)\mu. \qquad (2.46)
\end{aligned}
$$

Now, from (2.33), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40), (2.41), (2.43), and (2.46), and by defining $M_g > 0$ (depending on a bound for the second derivatives of $\theta_g(w)$) and $M_\theta = C_1 + C_2 + C_3 + C_4 + C_5 > 0$, we obtain

$$
\theta_g(w) - \theta_g(w(\Delta)) \geq M_\mu\alpha^t(\Delta)\mu - M_g\Delta^2 - M_\theta\theta(w),
$$

where

$$
M_\mu = n^2\left(\max\left\{1, \frac{1-\sigma}{\gamma}\right\}\right)^2.
$$

We have thus proved (2.32).

We can now easily show (2.34). In fact, from $m(w(\Delta)) = m(w) + \theta_g(w)^T s(\Delta)$, we have that

$$
m(w) - m(w(\Delta)) \geq M_\mu\alpha^t(\Delta)\mu - M_\theta\theta(w).
$$

□

We show next that for any $\epsilon > 0$ and all $p \in [1, 2]$, $q \in (1, 2]$ there exists a $\Delta_{\min}(\epsilon) > 0$ such that for any point $w \in \mathcal{N}(\gamma, M, p)$ with $\theta_\ell(w) + \mu \geq \epsilon$ we have also $w(\Delta) \in \mathcal{N}(\gamma, M, p)$ for all $0 < \Delta \leq \Delta_{\min}(\epsilon)$.

We need the following auxiliary result, which will be used several times in the convergence theory.

LEMMA 2.6. *If* $w \in \mathcal{N}(\gamma, M, p)$ *and* $\theta_\ell(w) + \mu \geq \epsilon$ *for some* $\epsilon > 0$ *then*

$$
\mu \geq \min\left\{\frac{\epsilon}{2}, \frac{(\epsilon/2)^p}{M}\right\} \stackrel{\text{def}}{=} a(\epsilon).
$$

Proof. Since $\theta_\ell(w) + \mu \geq \epsilon$, we have either $\mu \geq \epsilon/2$ or $\theta_\ell(w) \geq \epsilon/2$. In the second case, using the fact that $w \in \mathcal{N}(\gamma, M, p)$, we obtain, from $\theta_h(w) + \theta_\ell(w)^p \leq M\mu$, that

$$
\mu \geq \frac{\theta_\ell(w)^p}{M} \geq \frac{(\epsilon/2)^p}{M}.
$$

So, combining both cases, we have

$$\mu \geq \min\left\{\frac{\epsilon}{2}, \frac{(\epsilon/2)^p}{M}\right\}.$$

□

LEMMA 2.7. *Let* $\gamma \in (0,1)$*,* $M > 0$*,* $p, q \in (1,2]$ *and let* (2.21) *hold. Moreover, assume that* $w \in \mathcal{N}(\gamma, M, p)$*,* $\mathrm{KKT}(w)$ *is invertible, and* $\theta_\ell(w) + \mu \geq \epsilon$*. Then there exists a constant* $\Delta_{\min}(\epsilon)$ *dependent on an upper bound on* $\|\mathrm{KKT}(w)^{-1}\|$ *such that, if* $0 < \Delta \leq \Delta_{\min}(\epsilon)$*, then* $w(\Delta) \in \mathcal{N}(\gamma, M, p)$*.*

REMARK 2.1.

1. If the exact Hessian $H = \nabla^2_{xx}\ell(w)$ is used then Lemma 2.7 holds with some $\Delta_{\min} > 0$ not dependent on ϵ when $p = 1$ (see [18]). This behavior indicates that the neighborhood does not prevent fast local convergence.
2. By using Lemma 2.1.ii) it is possible to extend Lemma 2.7 to the case $p = 1$ if the condition $\theta_h(w) + \theta_\ell(w)^p \leq M\frac{x^T z}{n}$ in the definition of the neighborhood $\mathcal{N}(\gamma, M, p)$ is replaced by $\theta_h(w) \leq M\frac{x^T z}{n}$, $\theta_\ell(w)^p \leq M\frac{x^T z}{n}$. This modification in the definition of the central neighborhood would still retain the main global convergence properties.

Proof. By Lemma 2.6 we have

$$\mu \geq \min\left\{\frac{\epsilon}{2}, \frac{(\epsilon/2)^p}{M}\right\} \stackrel{\text{def}}{=} a(\epsilon).$$

The result will be proved for

$$\Delta_{\min}(\epsilon) = \min\left\{1, \sqrt{\frac{\sigma(1-\gamma)a(\epsilon)}{4(1+\gamma)}}, \frac{\sigma(1-\gamma)}{4(1+\gamma)C(M+n)}, \left(\frac{\sigma M a(\epsilon)}{M_h + M_\ell + 4M}\right)^{\frac{1}{\min\{p,q\}}}, \left(\frac{\sigma M b(\epsilon)}{M_h + M_\ell + 4M}\right)^{\frac{1}{\min\{p,q\}-1}}\right\}, \tag{2.47}$$

where C is an upper bound for $\|\mathrm{KKT}(w)^{-1}\|$ and

$$b(\epsilon) \stackrel{\text{def}}{=} \frac{1}{C(\max\{M, M^{\frac{1}{p}} a(\epsilon)^{\frac{1}{p}-1}\} + n)}.$$

1. We first show that $X(\Delta)z(\Delta) \geq \gamma\mu(\Delta)e$ holds for all $0 < \Delta \leq \Delta_{\min}(\epsilon)$ with $\Delta_{\min}(\epsilon)$ given in (2.47).

Following exactly the same steps as in the first part of the proof of [18, Lemma 2], we can claim that $X(\Delta)z(\Delta) \geq \gamma\mu(\Delta)e$ holds provided

$$\Delta \leq \min\left\{\sqrt{\frac{\sigma(1-\gamma)\mu}{4(1+\gamma)}}, \frac{\sigma(1-\gamma)}{4(1+\gamma)C(M+n)}\right\}. \tag{2.48}$$

The bound (2.48) holds since $\mu \geq a(\epsilon)$ and $\Delta \leq \Delta_{\min}(\epsilon)$. This part of the proof is exactly the same as in [18, Lemma 2], since it involves blocks of the primal-dual system unaffected by the possibility of having $H \neq \nabla^2_{xx}\ell(w)$.

2. We prove now that also the condition

$$\theta_h(w(\Delta)) + \theta_\ell(w(\Delta))^p \leq M\mu(\Delta) \quad \text{for all } 0 < \Delta \leq \Delta_{\min}(\epsilon) \tag{2.49}$$

holds with $\Delta_{\min}(\epsilon)$ defined in (2.47).

We first note that $\|s^n\| \leq C(M+n)\mu$ as in [18], but the upper bound on the norm of s^t is now — due to the changes in the definition of the central neighborhood — of the form

$$\|s^t\| \leq C((M\mu)^{\frac{1}{p}} + (1-\sigma)n^{\frac{1}{2}}\mu) \leq C(M\mu)^{\frac{1}{p}} + n\mu.$$

We obtain

$$\delta \stackrel{\text{def}}{=} \max\{\|s^n\|, \|s^t\|\} \leq C(\max\{M\mu, (M\mu)^{\frac{1}{p}}\} + n\mu). \tag{2.50}$$

From (2.22), (2.30), and $\alpha^t(\Delta) \leq \alpha^n(\Delta)$, we know that

$$\begin{aligned} \theta_\ell(w(\Delta))^p &\leq (1-\alpha^t(\Delta))\theta_\ell(w)^p + M_\ell \max\{\Delta^p, \Delta^q, \Delta^2\}, \\ \theta_h(w(\Delta)) &\leq (1-\alpha^t(\Delta))\theta_h(w) + M_h\Delta^2. \end{aligned}$$

Using $\theta_h(w) + \theta_\ell(w)^p \leq M\mu$ we get

$$\theta_h(w(\Delta)) + \theta_\ell(w(\Delta))^p \leq (1-\alpha^t(\Delta))M\mu + (M_h + M_\ell)\max\{\Delta^2, \Delta^q, \Delta^p\}.$$

On the other hand, by (2.29),

$$M\mu(\Delta) \geq (1-\alpha^t(\Delta))M\mu + \sigma\alpha^t(\Delta)M\mu - 4M\Delta^2.$$

Therefore, (2.49) holds whenever

$$(M_h + M_\ell + 4M)\max\{\Delta^2, \Delta^q, \Delta^p\} \leq \sigma\alpha^t(\Delta)M\mu.$$

Now we have $\Delta_{\min}(\epsilon) \leq 1$ and thus, from $p, q \in (1, 2]$, (2.49) is true when

$$(M_h + M_\ell + 4M)\Delta^{\min\{p,q\}} \leq \sigma\alpha^t(\Delta)M\mu,$$

which by (2.14) is implied by

$$\Delta \leq \min\left\{\left(\frac{\sigma M\mu}{M_h + M_\ell + 4M}\right)^{\frac{1}{\min\{p,q\}}}, \left(\frac{\sigma M\mu}{(M_h + M_\ell + 4M)\delta}\right)^{\frac{1}{\min\{p,q\}-1}}\right\}. \tag{2.51}$$

By using (2.50) we have

$$\frac{\mu}{\delta} \geq \frac{1}{C(\max\{M, M^{\frac{1}{p}}\mu^{\frac{1}{p}-1}\} + n)} \geq \frac{1}{C(\max\{M, M^{\frac{1}{p}}a(\epsilon)^{\frac{1}{p}-1}\} + n)} \stackrel{\text{def}}{=} b(\epsilon).$$

Therefore, (2.51) is implied by $\Delta \leq \Delta_{\min}(\epsilon)$ with $\Delta_{\min}(\epsilon)$ defined in (2.47).

3. The proof that $X(\Delta)z(\Delta) > 0$ holds for all Δ such that (2.47) is satisfied is exactly as in [18, Lemma 2]. □

3. The interior-point filter method. The filter entries in [18] were designed to meet the goal of reducing feasibility and centrality, combined by $\theta(w) = \theta_c(w) + \theta_h(w)$, and simultaneously complementarity and criticality, measured by $\mu + \theta_\ell(w)$. This approach differed from the traditional choices of the filter entries in filter methods in two respects: by adding centrality to feasibility in the first filter entry, and by replacing the objective function in the second filter entry by the optimality measure $\mu + \theta_\ell(w)$. In this paper, we use a similar approach, choosing $\theta(w)$ and $\theta_g(w)$ to form each filter entry, but where now $\theta_g(w)$ is given either by (2.11) or by (2.12). The difference thus consists in the second filter entry which now has a direct contribution of the objective function. The other important modification to the algorithm introduced in [18] is the possibility of using approximations H to the Hessian of the Lagrangian, provided they satisfy (2.21), and the appropriate adjustment in the definition of the central neighborhood in (2.18).

Before we describe the algorithm in detail we need to discuss and motivate a number of its components. We state first the definitions of dominance and filter used by the algorithm. A point w, or the corresponding pair $(\theta(w), \theta_g(w))$, is said to dominate a point w', or the corresponding pair $(\theta(w'), \theta_g(w'))$, if

$$\theta(w) \leq \theta(w') \quad \text{and} \quad \theta_g(w) \leq \theta_g(w'),$$

or, equivalently, if the following inequality is violated:

$$\max\{\theta(w) - \theta(w'), \theta_g(w) - \theta_g(w')\} > 0.$$

A filter is a finite subset $\mathcal{F} \subset \mathbb{R}^2$ consisting of pairs (θ^f, θ_g^f), with $\theta^f \stackrel{\text{def}}{=} \theta_h^f + \theta_c^f$, such that no pair dominates any of the others.

As it is well known in filter methods, the requirement that a new iterate is not dominated by any of the filter entries is not enough, and some form of sufficient acceptance is necessary. We choose to work with an envelope type filter. Let $\gamma_{\mathcal{F}} \in (0, 1/2)$ be fixed. The point w is acceptable to the filter $\mathcal{F}$ if, for all $(\theta^f, \theta_g^f) \in \mathcal{F}$, it holds

$$\max\{\theta^f - \theta(w), \theta_g^f - \theta_g(w)\} > \gamma_{\mathcal{F}} \theta^f.$$

The procedure to add entries to the filter is summarized below. Note that if w is added to the filter, all previous entries that are dominated by the new entry are removed. By adding w to the filter $\mathcal{F}$ we mean the following operation:

$$\begin{aligned} \mathcal{F} \mapsto \mathcal{F} = &\{(\theta(w), \theta_g(w))\} \cup \\ &\big\{(\theta^f, \theta_g^f) \in \mathcal{F} \ : \ \min\{\theta^f - \theta(w), \theta_g^f - \theta_g(w)\} < 0\big\}. \end{aligned}$$

Our primal-dual interior-point filter method generates iterates $w_{k+1} = w_k(\Delta_k) \neq w_k$ acceptable to the filter. Not all new iterates w_{k+1} are, however, added to the filter.

In general, the primal-dual interior-point filter method imposes a sufficient reduction criterion relating the actual reduction in θ_g with the reduction predicted by its model m_k, of the form $\rho_k \geq \eta$ where

$$\rho_k \stackrel{\text{def}}{=} \frac{\theta_g(w_k) - \theta_g(w_k(\Delta_k))}{m_k(w_k) - m_k(w_k(\Delta_k))}$$

and $\eta \in (0, 1)$ is a preset constant. However, the test of this condition is skipped if

$$m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa \theta(w_k)^2,$$

where $\kappa \in (0, 1)$ is a preset constant. In other words, the sufficient reduction criterion $\rho_k \geq \eta$ is only imposed when the reduction in the model m_k is sufficiently good compared with $\theta(w_k)^2$. In the situation where $\rho_k < \eta$ and $m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa \theta(w_k)^2$, the new iterate $w_{k+1} = w_k(\Delta_k)$ is accepted and the previous point w_k is added to the filter (guaranteeing that this new filter entry satisfies $\theta(w_k) > 0$). This criterion for adding points w_k to the filter prevents us from building up a filter for which the computation of acceptable points would require too small trust-region radii.

If $\rho_k \geq \eta$ and $m_k(w_k) - m_k(w_k(\Delta_k)) \geq \kappa\theta(w_k)^2$, the iterate w_k is not added to the filter. This situation is the only one where a new iterate $w_{k+1} = w_k(\Delta_k)$ is computed and the previous one, w_k, is not added to the filter.

If $\theta(w_k)$ is too large compared to Δ_k (or an appropriate power of Δ_k), the algorithm enters a restoration phase with the purpose of reducing θ. More precisely, a restoration algorithm is called if

$$\theta(w_k) > \Delta_k \min\{\gamma_1, \gamma_2 \Delta_k^\beta\},$$

where γ_1, γ_2, and β are preset positive constants. The restoration algorithm must produce a new iterate w_{k+1} that is not only acceptable to the filter but also satisfies $\theta(w_{k+1}) \leq \Delta_k \min\{\gamma_1, \gamma_2 \Delta_k^\beta\}$. In this situation, the previous iterate w_k is added to the filter (guaranteeing also that this new filter entry also satisfies $\theta(w_k) > 0$).

The new primal-dual interior-point filter method satisfying the above features can now be presented (see Algorithm 3.1 and Figure 3.1). Note that step 5 guarantees that the potentially new iterate $w_k(\Delta_k)$ is always acceptable to the filter. In the following algorithm, the current iterate in iteration k is denoted by w_k and the normal and tangential trial steps are denoted by s_k^t and s_k^n, respectively. Further, the step sizes $\alpha_k^n(\Delta)$ and $\alpha_k^t(\Delta)$ are defined according to (2.13) and (2.14), respectively, with $s^n = s_k^n$ and $s^t = s_k^t$. Similarly, $w_k(\Delta)$ and $s_k(\Delta)$ are defined by (2.15) and (2.16), respectively, with $w = w_k$, $s^{n/t} = s_k^{n/t}$, and $\alpha^{n/t}(\Delta) = \alpha_k^{n/t}(\Delta)$.

ALGORITHM 3.1 (Primal-dual interior-point filter method).

0. Choose $\sigma \in (0,1)$, $\nu \in (0,1)$ $\gamma_1, \gamma_2 > 0$, $0 < \beta, \eta, \kappa < 1$, $\gamma_{\mathcal{F}} \in (0, 1/2)$, and $p \in [1,2]$. Set $\mathcal{F} := \emptyset$. Choose $(x_0, z_0) > 0$ and y_0, and determine $\gamma \in (0,1)$ such that $X_0 z_0 \geq \gamma\mu_0$ with $\mu_0 = x_0^T z_0/n$. Further, choose $M > 0$ such that $\theta_h(w_0) + \theta_\ell(w_0)^p \leq M\mu_0$. Choose $\Delta_0^{in} > 0$ and set $k := 0$.
1. Set $\mu_k := x_k^T z_k/n$ and compute s_k^n and s_k^t by solving the linear systems (2.8) and (2.9), respectively, with $(w, \mu) = (w_k, \mu_k)$.
2. Compute $\Delta_k' \in [0, \Delta_k^{in}]$ such that

   $$x_k(\Delta) > 0, \quad z_k(\Delta) > 0, \quad X_k(\Delta) z_k(\Delta) \geq \gamma\mu_k(\Delta)e \quad \text{for all } \Delta \in [0, \Delta_k']$$

   and such that Δ_k' is not smaller than the largest $\nu^r \Delta_k^{in}$, $r = 0, 1, \ldots$, having this property.
3. Compute the largest $\Delta_k'' = \nu^j \Delta_k'$, $j = 0, 1, \ldots$, such that

   $$\theta_h(w_k(\Delta_k'')) + \theta_\ell(w_k(\Delta_k''))^p \leq M\mu_k(\Delta_k'').$$

   Set $\Delta_k := \Delta_k''$.
4. If $\theta(w_k) \leq \Delta_k \min\{\gamma_1, \gamma_2 \Delta_k^\beta\}$ then continue in step 5. Otherwise **add** w_k to the filter and use a restoration algorithm to produce a point w_{k+1} such that:

   $w_{k+1} \in \mathcal{N}(\gamma, M, p)$ with $\mu_{k+1} = x_{k+1}^T z_{k+1}/n$;
   w_{k+1} is acceptable to the filter;
   $\theta(w_{k+1}) \leq \Delta_{k+1}^{in} \min\{\gamma_1, \gamma_2 (\Delta_{k+1}^{in})^\beta\}$ with $\Delta_{k+1}^{in} = \Delta_k$.

   Set $\Delta_{k+1}^{in} := \Delta_k$, $k := k+1$, and go to step 1.
5. If $w_k(\Delta_k)$ is not acceptable to the filter (with w_k considered in the filter if $m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa\theta(w_k)^2$), then go to step 11.
6. If $m_k(w_k) - m_k(w_k(\Delta_k)) = 0$, then set $\rho_k := 0$. Otherwise, compute

   $$\rho_k = \frac{\theta_g(w_k) - \theta_g(w_k(\Delta_k))}{m_k(w_k) - m_k(w_k(\Delta_k))}.$$

7. If $\rho_k < \eta$ and $m_k(w_k) - m_k(w_k(\Delta_k)) \geq \kappa\theta(w_k)^2$ then go to step 11.
8. If $m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa\theta(w_k)^2$ then **add** w_k to the filter.
9. Choose $\Delta_{k+1}^{in} \geq \Delta_k$.
10. Set $w_{k+1} := w_k(\Delta_k)$, $k := k+1$, and go to step 1.
11. Set $w_{k+1} := w_k$, $s_{k+1}^n := s_k^n$, $s_{k+1}^t := s_k^t$, $\Delta'_{k+1} := \Delta_k/2$, and $\Delta_{k+1}^{in} := \Delta'_{k+1}$. Set $k := k+1$ and go to step 3.

In practice, step 2 would be implemented as $\Delta'_k = \tau_k \hat{\Delta}'_k$, where $\hat{\Delta}'_k$ is the largest value of Δ such that $(x_k(\Delta), z_k(\Delta)) \geq 0$ and $X_k(\Delta)z_k(\Delta) \geq \gamma\mu_k(\Delta)e$ and τ_k is a parameter in $(\nu, 1)$ to enforce $(x_k(\Delta), z_k(\Delta)) > 0$. The adjustment of τ_k would be important to achieve a rapid rate of local convergence. We point out that the calculation of Δ_k is split in steps 2 and 3 for good reasons. In fact, in step 2 it is possible to determine explicitly Δ'_k (more precisely $\hat{\Delta}'_k$). However, because of the nonlinearity of θ_h and θ_ℓ, that is not the case in step 3, where we know from Lemma 2.7 that although there exists a sufficiently small Δ''_k satisfying $\theta_h(w_k(\Delta''_k)) + \theta_\ell(w_k(\Delta''_k))^p \leq M\mu_k(\Delta''_k)$, it cannot be determined explicitly.

In practice, step 1 of the algorithm would start by checking the satisfaction of a stopping criterion of the form $\theta(w_k) + \theta_\ell(w_k) + \mu_k \leq \epsilon$, for small $\epsilon > 0$. To be able to analyze the asymptotic global convergence properties of the algorithm we did not include any stopping criterion.

4. Global convergence to first-order critical points. We will assume that the functions f and h defining problem (2.1) and the sequence of iterates $\{w_k\}$ generated by the primal-dual interior-point filter method (Algorithm 3.1) satisfy the following set of assumptions.

ASSUMPTION 4.1.

- (A1) The sequence $\{(x_k, y_k, z_k)\}$ is bounded.
- (A2) The derivatives ∇h and $\nabla^2_{xw}\ell$ exist and are Lipschitz continuous in an open set containing all the iterates (x_k, y_k, z_k) and the line segments $[w_k, w_k + s_k(\Delta_k)]$.
- (A3) The sequence $\{H_k\}$ is bounded. The choice of the Hessian H_k must allow the satisfaction of the sufficient decrease condition (2.22), e.g., should satisfy the condition (2.21).
- (A4) The matrix $H_k + \frac{1}{2}X_k^{-1}Z_k$ is positive semidefinite on the null space of $\nabla h(x_k)^T$ for all k.
- (A5) There exists $C > 0$ such that for all k it holds $\|\,\mathrm{KKT}(w_k)^{-1}\| \leq C$.

Assumptions (A1)-(A3) are standard in the globalization of algorithms for nonlinear programming. Assumption (A4) is satisfied, for instance, for all positive semidefinite choices of H_k. Moreover, Assumption (A4) holds also in a neighborhood of a regular point w^* satisfying the second-order sufficient conditions and strict complementarity, for the particular choice $H_k = \nabla^2_{xx}\ell(w_k)$. Therefore, Assumption (A4) can be ensured globally by choosing H_k sufficiently positive definite and is implied locally by standard second-order sufficiency conditions.

These assumptions will allow us to prove global convergence to KKT points, which, of course, are feasible. Thus, this set of assumptions restrict in some form the analysis to problems that are not infeasible. As pointed out in [18], it is the uniform boundedness of the $\mathrm{KKT}(w_k)^{-1}$ assumed in Assumption (A.5) that is responsible for ruling out infeasibility.

We will prove in this paper the following global convergence result.

THEOREM 4.1. *Under Assumption 4.1, the sequence of iterates $\{w_k\}$ generated by the primal-dual interior-point method (Algorithm 3.1) satisfies*

$$\liminf_{k\to\infty}\ \theta(w_k) + \theta_\ell(w_k) + \mu_k = 0.$$

The proof of Theorem 4.1 requires the adaption of the analysis of global convergence of [18] to the new optimality filter entries (2.11) and (2.12) introduced in this paper.

The first result is a direct consequence of Assumptions 4.1 and of Lemmas 2.4, 2.5 and 2.7.

LEMMA 4.2. *The following hold:*

i) *The sequences $\{\theta_h(w_k)\}$, $\{\theta_c(w_k)\}$, $\{\theta_\ell(w_k)\}$, $\{\mu_k\}$, and $\{\theta_g(w_k)\}$ are bounded.*

ii) *The constants M_h, M_c, M_μ, M_g, and M_θ in Lemma 2.4 and Lemma 2.5 are bounded for all k.*

iii) *For all $\epsilon > 0$, there exists a positive constant $\Delta_{\min}(\epsilon)$ such that, for all k for which $\theta_\ell(w_k) + \mu_k \geq \epsilon$, the conditions in steps 2 and 3 are satisfied for all $\Delta_k', \Delta_k'' \in [0, \Delta_{\min}(\epsilon)]$. Thus, steps 2 and 3 leave Δ_k^{in} unchanged for $0 \leq \Delta_k^{in} \leq \Delta_{\min}(\epsilon)$ and we have $\Delta_k = \Delta_k^{in}$.*

iv) *It holds that $\|s_k^n\| \leq C(M + (n^2 - n)^{1/2})\mu_k$ and $\|s_k^t\| \leq C((M\mu_k)^{\frac{1}{p}} + (1 - \sigma)n^{1/2}\mu_k)$ for all k.*

As in [18], note that the result iv) follows from $\|Xz - \mu e\| \leq (n^2 - n)^{1/2}\mu$ and $\|(1 - \sigma)\mu e\| \leq (1 - \sigma)n^{1/2}\mu$. Given the fact that $\{(x_k, y_k, z_k)\}$ is bounded and α_k^n and α_k^t do not exceed one, one concludes from Lemma 4.2.iv that the sequence $\{s(\Delta_k)\}$ is also bounded.

We show in the next lemma, as a direct consequence of the mechanisms of the algorithm, that the first components of all filter entries are positive.

LEMMA 4.3. *If w_k is added to the filter, then $\theta(w_k) > 0$.*

Proof. An iterate w_k is added to the filter either in step 4 or in step 8. In the first case (step 4), we see that $\theta(w_k) > \Delta_k \min\{\gamma_1, \gamma_2 \Delta_k^\beta\} > 0$. In the second case (step 8), we see from Lemma 2.5, (2.34) that

$$\theta(w_k)^2 > \frac{1}{\kappa}(m_k(w_k) - m_k(w_k(\Delta_k)) \geq \frac{1}{\kappa}(M_\mu \alpha_k^t \mu_k - M_\theta \theta(w_k)).$$

So, by contradiction, if we assume $\theta(w_k) = 0$, we would get

$$0 = \theta(w_k)^2 > \frac{1}{\kappa}(m_k(w_k) - m_k(w_k(\Delta_k)) \geq \frac{1}{\kappa} M_\mu \alpha^t \mu_k \geq 0.$$

Thus, in both cases, $\theta(w_k) > 0$. □

It also requires no analysis and follows directly from the structure of the algorithm that new iterates are always acceptable to the filter.

LEMMA 4.4. *In all iterations $k \geq 0$, the current iterate w_k is acceptable to the filter.*

Proof. See [18, Lemma 5]. □

The next four lemmas provide some technical results needed to establish global convergence to first-order critical points. The first of these lemmas establishes a crucial inequality showing that feasibility and centrality at $w_k(\Delta_k)$ are of the order of Δ_k^2.

LEMMA 4.5. *There exists a $\Delta_r > 0$ such that, if $\Delta_k \leq \Delta_r$ in step 5, it holds that*

$$\theta(w_k(\Delta_k)) \leq (M_h + M_c)\Delta_k^2.$$

Proof. See [18, Lemma 6]. □

It is important to note that Lemma 4.5 (and the next two Lemmas 4.6 and 4.7) deal with the situation in step 5 of the algorithm. Step 5 is preceded by step 4, and thus, in step 5 it always holds that

$$\theta(w_k) \leq \Delta_k \min\{\gamma_1, \gamma_2 \Delta_k^\beta\}, \tag{4.1}$$

since otherwise step 4 calls restoration instead of step 5. We point out that it is (4.1) that enables to show $\theta(w_k(\Delta_k)) = \mathcal{O}(\Delta_k^2)$ for sufficiently small values of Δ_k in Lemma 4.5.

The next two lemmas establish that $w_k(\Delta_k)$ is acceptable to the filter in step 5 for sufficiently small values of Δ_k. The results are similar to those proved in [18]. In both lemmas we analyze the acceptability of $w_k(\Delta_k)$ to the filter with w_k considered in the filter if $m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa\theta(w_k)^2$. The latter is needed since, in this situation, w_k will possibly be added to the filter in step 8. First, we derive a result that depends on the current filter entries.

LEMMA 4.6. *Suppose that $\theta(w_k) + \theta_\ell(w_k) + \mu_k \geq \epsilon > 0$. Then there exists $\Delta_a(\epsilon) > 0$ depending on ϵ and on the values of the filter entries, such that, if*

$$0 < \Delta_k \leq \Delta_a(\epsilon),$$

then $w_k(\Delta_k)$ is acceptable to the filter in step 5 (with w_k considered in the filter when $m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa\theta(w_k)^2$).

Proof. Since $0 < \gamma_{\mathcal{F}} < 1/2 < 1$, we have from Lemma 4.3 that

$$\theta_{\mathcal{F}} \stackrel{\text{def}}{=} \min_{(\theta^f, theta_g^f) \in \mathcal{F}} (1 - \gamma_{\mathcal{F}})\theta^f > 0.$$

Consider first the case where $\theta(w_k) \geq \epsilon/2$. Then $w_k(\Delta_k)$ is acceptable to the filter (with w_k considered in the filter when $m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa\theta(w_k)^2$) if

$$\theta(w_k(\Delta_k)) \leq \frac{1}{2}\min\{\theta_{\mathcal{F}}, (1 - \gamma_{\mathcal{F}})\epsilon/2\} < \min\{\theta_{\mathcal{F}}, (1 - \gamma_{\mathcal{F}})\epsilon/2\}. \qquad (4.2)$$

We also know from Lemma 4.5 that

$$\theta(w_k(\Delta_k)) \leq (M_h + M_c)\Delta_k^2$$

holds for $\Delta_k \leq \Delta_r$. Thus, (4.2) is satisfied for $\Delta_k \leq \Delta_a^{(1)}(\epsilon)$ with $\Delta_a^{(1)}(\epsilon) > 0$ depending only on $\theta_{\mathcal{F}}$, ϵ, M_h, M_c, $\gamma_{\mathcal{F}}$, and Δ_r.

Otherwise we have $\theta_\ell(w_k) + \mu_k \geq \epsilon/2$. Thus, Lemma 2.6 yields

$$\mu_k \geq a(\epsilon/2).$$

If w_k is not considered in the filter in step 5, then a similar argument, with $\theta(w_k(\Delta_k)) \leq \frac{1}{2}\theta_{\mathcal{F}}$ instead of (4.2), shows that if $\Delta_k \leq \Delta_a^{(1)}(\epsilon)$ then $w_k(\Delta_k)$ is acceptable to the filter. Moreover $w_k(\Delta_k)$ is also acceptable, with w_k considered in the filter when $m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa\theta(w_k)^2$, if, in addition,

$$\theta_g(w_k(\Delta_k)) - \theta_g(w_k) < -\gamma_{\mathcal{F}}\theta(w_k). \qquad (4.3)$$

In the rest of the proof we show how this bound can be achieved for sufficiently small Δ_k. Since step 5 is reached, we know that

$$\theta(w_k) \leq \gamma_2\Delta_k^{1+\beta}.$$

On the other hand, we obtain from $\mu_k \geq a(\epsilon/2)$ and Lemma 2.5 that

$$\begin{aligned}\theta_g(w_k(\Delta_k)) - \theta_g(w_k) &\leq -M_\mu\alpha_k^t a(\epsilon/2) + M_\theta\theta(w_k) + M_g\Delta_k^2 \\ &\leq -M_\mu\alpha_k^t a(\epsilon/2) + M_\theta\gamma_2\Delta_k^{1+\beta} + M_g\Delta_k^2.\end{aligned}$$

Hence it is sufficient to show that

$$-M_\mu \alpha_k^t a(\epsilon/2) + M_g \Delta_k^2 < -(\gamma_{\mathcal{F}} + M_\theta)\gamma_2 \Delta_k^{1+\beta}.$$

Since $\|s_k^n\|$ and $\|s_k^t\|$ are bounded by a constant M_s and $\alpha_k^t = \min\{1, \frac{\Delta_k}{\|s_k^n\|}, \frac{\Delta_k}{\|s_k^t\|}\}$, we have for all $\Delta_k \leq M_s$, that $\alpha_k^t \geq \Delta_k/M_s$. Thus (4.3) holds if

$$M_g \Delta_k + (\gamma_{\mathcal{F}} + M_\theta)\gamma_2 \Delta_k^\beta \leq \frac{M_\mu a(\epsilon/2)}{2M_s} < \frac{M_\mu a(\epsilon/2)}{M_s},$$

which in turn holds for all $\Delta_k \leq \Delta_a^{(2)}(\epsilon)$ with $\Delta_a^{(2)}(\epsilon) > 0$ depending only on ϵ, M_g, M_θ, $\gamma_{\mathcal{F}}$, γ_2, β, M_μ, $a(\epsilon/2)$, and M_s. Taking $\Delta_a(\epsilon) = \min\{\Delta_a^{(1)}(\epsilon), \Delta_a^{(2)}(\epsilon)\}$ concludes the proof. □

Now we derive a similar result that does not depend on the values of the filter entries, but where we impose a condition relating $\theta(w_k)$ and Δ_k.

LEMMA 4.7. *Suppose that for given* $\epsilon > 0$

$$\theta_\ell(w_k) + \mu_k \geq \epsilon \quad \textit{and} \quad \theta(w_k) > \frac{\Delta_k}{2}\min\{\gamma_1, \gamma_2(\Delta_k/2)^\beta\}. \tag{4.4}$$

Then there exists $\Delta_f(\epsilon) > 0$ *depending on* ϵ*, but not on the filter entries, such that, if*

$$0 < \Delta_k \leq \Delta_f(\epsilon),$$

then $w_k(\Delta_k)$ *is acceptable to the filter in step 5 (with* w_k *considered in the filter when* $m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa\theta(w_k)^2$*).*

Proof. Since, by Lemma 4.4, w_k is acceptable to the filter, then $w_k(\Delta_k)$ is acceptable to the filter (with w_k considered in the filter when $m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa\theta(w_k)^2$) if

$$\theta(w_k(\Delta_k)) \leq \theta(w_k)$$

and

$$\theta_g(w_k(\Delta_k)) < \theta_g(w_k) - \gamma_{\mathcal{F}}\theta(w_k). \tag{4.5}$$

We know from Lemma 4.5 that, if $\Delta_k \leq \Delta_r$ then

$$\theta(w_k(\Delta_k)) \leq (M_h + M_c)\Delta_k^2.$$

Hence, $\theta(w_k(\Delta_k)) \leq \theta(w_k)$ is ensured by the second inequality in (4.4) if in addition

$$(M_h + M_c)\Delta_k \leq \frac{1}{2}\min\{\gamma_1, \gamma_2(\Delta_k/2)^\beta\}. \tag{4.6}$$

Moreover, the first inequality in (4.4) and Lemma 2.6 yield

$$\mu_k \geq a(\epsilon).$$

Therefore, we have by Lemma 2.5

$$\theta_g(w_k(\Delta_k)) - \theta_g(w_k) \leq -M_\mu \alpha_k^t a(\epsilon) + M_\theta\theta(w_k) + M_g\Delta_k^2.$$

We have pointed out before that $\alpha_k^t \geq \Delta_k/M_s$ for all $\Delta_k \leq M_s$, see the end of the proof of Lemma 4.6. So,

$$\theta_g(w_k(\Delta_k)) - \theta_g(w_k) \leq \Delta_k\left(-\frac{M_\mu a(\epsilon)}{M_s} + M_g\Delta_k\right) + M_\theta\theta(w_k).$$

Since we are concerned with step 5 of the algorithm, we know that $\theta(w_k) \leq \gamma_2 \Delta_k^{1+\beta}$, see (4.1). Hence, we obtain (4.5) whenever

$$M_g \Delta_k + (\gamma_{\mathcal{F}} + M_\theta)\gamma_2 \Delta_k^\beta \leq \frac{M_\mu a(\epsilon)}{2M_s} < \frac{M_\mu a(\epsilon)}{M_s}. \tag{4.7}$$

The requirements $0 < \Delta_k \leq \Delta_r$, (4.6) and (4.7) on Δ_k are obviously satisfied if $0 < \Delta_k \leq \Delta_f(\epsilon)$ with some constant $\Delta_f(\epsilon) > 0$. □

Now we are ready to derive asymptotic results. We appeal first to a commonly used argument in filter convergence proofs to show that $\liminf_{k\to\infty} \theta(w_k) = 0$ when infinitely many iterates are added to the filter.

LEMMA 4.8. *From the moment that w_k is added to the filter, the filter always contains an entry that dominates w_k.*

Proof. See [18, Lemma 9]. □

LEMMA 4.9. *Suppose there are infinitely many points added to the filter. Then there exists a subsequence $\{k_i\}$ such that w_{k_i} is added to the filter and*

$$\lim_{i\to\infty} \theta(w_{k_i}) = 0. \tag{4.8}$$

Proof. See [18, Lemma 10]. □

As we pointed out in [18], Lemma 4.9 asserts (4.8) only for some particular subsequence $\{w_{k_i}\}$ of iterates added to the filter and not for any such subsequence. The reason is that acceptability to a pair does not imply acceptability to a dominated pair. If required, this effect can be circumvented in several ways. The easiest approach is to never remove dominated entries from the filter. Then the above proof can be easily modified to establish that (4.8) holds for any infinite subsequence of iterates that are added to the filter. An alternative to derive this stronger result, if one wishes to remove dominated filter entries, can also be obtained by slightly modifying the filter acceptance test, see [5] and [6, §15.5]. In fact, if we require

$$\text{either} \quad \theta^f - \theta(w) > \gamma_{\mathcal{F}}\theta^f \quad \text{or} \quad \theta_g^f - \theta_g(w) > \gamma_{\mathcal{F}}\theta(w),$$

then acceptability to a pair implies acceptability to all dominated pairs and it is straightforward to prove that (4.8) holds for any infinite subsequence of iterates added to the filter, see [6, Lem. 15.5.2].

We now proceed with the analysis. The next step is to show that when there are infinitely many iterates added to the filter, then the sequence $\{k_i\}$ of Lemma 4.9 must contain a subsequence converging to a first-order KKT point. Note that the subsequence $\{k_i\}$ must contain either a subsequence where restoration is invoked, or a subsequence where the iterates are added to the filter in step 8. We consider the two cases separately in the Lemmas 4.10 and 4.11. We start by considering an infinite number of iterations in $\{k_i\}$ at which restoration is invoked.

LEMMA 4.10. *Suppose that there exists an infinite sequence $\{k_i\}$ of iterations at which restoration is invoked and for which holds that*

$$\lim_{i\to\infty} \theta(w_{k_i}) = 0.$$

Then $\{k_i\}$ contains a subsequence $\{k'_j\}$ with

$$\lim_{j\to\infty} \theta(w_{k'_j}) = 0, \qquad \lim_{j\to\infty} \theta_\ell(w_{k'_j}) + \mu_{k'_j} = 0.$$

Proof. Let k_i be a subsequence where restoration is invoked for every k_i (and thus w_{k_i} is added to the filter) such that $\lim_{i\to\infty} \theta(w_{k_i}) = 0$. For deriving a contradiction, assume that there exists $\epsilon > 0$ with

$$\theta_\ell(w_{k_i}) + \mu_{k_i} \ge \epsilon \quad \forall\, i.$$

By Lemma 2.6 this implies that

$$\mu_{k_i} \ge a(\epsilon) \stackrel{\text{def}}{=} \epsilon^* > 0 \quad \forall\, i.$$

Since the restoration is invoked it must hold that

$$\theta_{k_i} > \Delta_{k_i} \min\{\gamma_1, \gamma_2 \Delta_{k_i}^{\beta}\}. \tag{4.9}$$

Therefore, we have

$$0 = \lim_{i\to\infty} \theta_{k_i} = \lim_{i\to\infty} \Delta_{k_i}$$

and thus we can find $K_0 > 0$ such that $\Delta_{k_i} < \nu\Delta_{\min}(\epsilon_*)$ for all $k_i \ge K_0$ with $\Delta_{\min}(\epsilon_*)$ (depending on ϵ_*) from Lemma 4.2, iii), with $\nu \in (0, 1)$. We show next that

$$\Delta_{k_i-1} \le 2\Delta_{k_i}, \quad \Delta_{k_i} = \Delta_{k_i}^{in} \quad \text{for all } k_i \ge K_0, \tag{4.10}$$

which then yields

$$0 = \lim_{i\to\infty} \theta_{k_i} = \lim_{i\to\infty} \Delta_{k_i} = \lim_{i\to\infty} \Delta_{k_i-1}. \tag{4.11}$$

In fact, $\Delta_{k_i} < \nu\Delta_{\min}(\epsilon_*)$ for $k_i \ge K_0$ shows that $\Delta_{k_i} = \Delta_{k_i}^{in}$ for $k_i \ge K_0$, since, by Lemma 4.2, iii), step 2 and step 3 yield only $\Delta_{k_i} \ne \Delta_{k_i}^{in}$ if $\Delta_{k_i}^{in} > \Delta_{\min}(\epsilon_*)$. But then the result of step 2 and step 3 would be a radius $\Delta_{k_i} > \nu\Delta_{\min}(\epsilon_*)$, which is not the case for $k_i \ge K_0$. Thus, we have $\Delta_{k_i} = \Delta_{k_i}^{in}$ for $k_i \ge K_0$ and conclude that $\Delta_{k_i} \ge \Delta_{k_i-1}/2$ for all $k_i \ge K_0$. Thus, (4.10) and (4.11) holds.

We show next that there exists $K_1 \ge K_0 - 1$ such that

$$\mu_{k_i-1} \ge \epsilon_*/2 \quad \text{for all } k_i - 1 \ge K_1. \tag{4.12}$$

In fact, we have either $w_{k_i} = w_{k_i-1}$ or $w_{k_i} = w_{k_i-1}(\Delta_{k_i-1})$. In the first case (4.12) is obvious since then $\mu_{k_i-1} = \mu_{k_i} \ge \epsilon_*$, for all $k_i \ge K_0$ with $w_{k_i} = w_{k_i-1}$. In the case $w_{k_i} = w_{k_i-1}(\Delta_{k_i-1})$ it follows from Lemma 2.4 that

$$\mu_{k_i} = \mu_{k_i-1}(\Delta_{k_i-1}) \le (1 - \alpha_{k_i-1}^t(1-\sigma))\mu_{k_i-1} + 4\Delta_{k_i-1}^2,$$

and thus

$$\epsilon_* \le \mu_{k_i} \le \mu_{k_i-1} + 4\Delta_{k_i-1}^2.$$

We can therefore conclude from (4.11) that (4.12) holds for $K_1 \ge K_0 - 1$ large enough. Using (4.12) and applying Lemma 4.2, iii) with $\epsilon = \epsilon_*/2$, we can apply the same argument as for deriving (4.10) to show that there exists $K_2 \ge K_1$ with $\Delta_{k_i-1} = \Delta_{k_i-1}^{in}$ for all $k_i - 1 \ge K_2$. Hence, together with (4.10), this yields

$$\Delta_{k_i-1} = \Delta_{k_i-1}^{in}, \quad \Delta_{k_i} = \Delta_{k_i}^{in} \quad \text{for all } k_i - 1 \ge K_2. \tag{4.13}$$

We show next that step 5 is reached in all iterations $k_i - 1 \geq K_2$. In fact, otherwise the restoration procedure is called in iteration $k_i - 1$. Thus, we have $\Delta_{k_i}^{in} = \Delta_{k_i-1}$ and consequently $\Delta_{k_i} = \Delta_{k_i-1}$ by (4.13). Since by our assumption the restoration is invoked in iteration $k_i - 1$, by using $\Delta_{k_i} = \Delta_{k_i-1}$ the outcome of the restoration is an iterate w_{k_i} with

$$\theta_{k_i} \leq \Delta_{k_i-1} \min\{\gamma_1, \gamma_2 \Delta_{k_i-1}^{\beta}\} = \Delta_{k_i} \min\{\gamma_1, \gamma_2 \Delta_{k_i}^{\beta}\},$$

which contradicts (4.9). Hence, step 5 is reached for all iterations $k_i - 1 \geq K_2$ and thus in particular

$$\theta_{k_i-1} \leq \Delta_{k_i-1} \min\{\gamma_1, \gamma_2 \Delta_{k_i-1}^{\beta}\}. \tag{4.14}$$

Next, we show that step 7 must be reached for all iterations $k_i - 1$ with $k_i - 1 \geq K_3$ and $K_3 \geq K_2$ large enough. In fact, let $\Delta_f(\epsilon_*/2)$ be the bound of Lemma 4.7 corresponding to $\epsilon = \epsilon_*/2$ instead of ϵ. By (4.11) we can find $K_3 \geq K_2$ such that $\Delta_{k_i-1} \leq \Delta_f(\epsilon_*/2)$ holds for all $k_i - 1 \geq K_3$. Now assume that step 7 is not reached in iteration $k_i - 1 \geq K_3 - 1$. Then step 5 is followed by step 11 and thus $\theta_{k_i} = \theta_{k_i-1}$, and, using (4.13), $\Delta_{k_i} = \Delta_{k_i}^{in} = \Delta_{k_i-1}/2$. Therefore, by (4.9),

$$\theta_{k_i-1} > \frac{\Delta_{k_i-1}}{2} \min\{\gamma_1, \gamma_2 (\Delta_{k_i-1}/2)^{\beta}\}.$$

Hence, we obtain from Lemma 4.7 and (4.12) that $w_{k_i-1}(\Delta_{k_i-1})$ was acceptable to the filter in step 5, since $k_i - 1 \geq K_3$ ensures $\Delta_{k_i-1} \leq \Delta_f(\epsilon_*/2)$. Therefore, step 5 would not have branched to step 11 as assumed. Hence, step 7 is always reached in all iterations $k_i - 1 \geq K_3$.

We conclude the proof by showing the existence of $K_4 \geq K_3$ such that step 9 is reached for all iterations $k_i - 1$ with $k_i - 1 \geq K_4$. This will provide the desired contradiction: In fact, by (4.13) and steps 9, 10 we have $\Delta_{k_i} = \Delta_{k_i}^{in} \geq \Delta_{k_i-1}$, $w_{k_i} = w_{k_i-1}(\Delta_{k_i-1})$. Thus, since step 9 is reached via step 5, we can apply Lemma 4.5 to obtain for all $\Delta_{k_i-1} \leq \Delta_r$ (which holds by (4.11) for all i large enough)

$$\theta_{k_i} = \theta(w_{k_i-1}(\Delta_{k_i-1})) \leq (M_h + M_c)\Delta_{k_i-1}^2 \leq (M_h + M_c)\Delta_{k_i}^2.$$

This contradicts (4.9) and (4.11).

Hence, it remains to show that step 9 is eventually reached in all iterations $k_i - 1$ with $k_i - 1 \geq K_4$, $K_4 \geq K_3$ large enough. We recall that by (4.12) we have $\mu_{k_i-1} \geq \epsilon_*/2$. Now, from Lemma 2.5, (4.11), and (4.14), we obtain, for all $k_i \geq K_4$

$$\begin{aligned} m_{k_i-1}(w_{k_i-1}) - m_{k_i-1}(w_{k_i-1}(\Delta_{k_i-1})) \geq & \\ & \geq M_\mu \alpha_{k_i-1}^t \frac{\epsilon_*}{2} - M_\theta \theta(w_{k_i-1}) \\ & \geq \Delta_{k_i-1} M_\mu \frac{\epsilon_*}{2M_s} - M_\theta \gamma_2 \Delta_{k_i-1}^{1+\beta} \\ & \geq \Delta_{k_i-1} M_\mu \frac{\epsilon_*}{4M_s}, \end{aligned}$$

for all $k_i - 1 \geq K_4 \geq K_3$ with some $K_4 \geq K_3$. As before, M_s denotes an upper bound for $\|s_k^n\|$ and $\|s_k^t\|$. Also, we used again the fact that $\alpha_{k_i-1}^t \geq \Delta_{k_i-1}/M_s$ if $\Delta_{k_i-1} \leq M_s$, which holds by (4.11) possibly after increasing K_4. On the other hand, we have

$$\begin{aligned} &|m_{k_i-1}(w_{k_i-1}) - m_{k_i-1}(w_{k_i-1}(\Delta_{k_i-1})) - \theta_g(w_{k_i-1}) + \theta_g(w_{k_i-1}(\Delta_{k_i-1}))| \\ &\qquad = O(\Delta_{k_i-1}^2). \end{aligned}$$

The last two bounds show that $\rho_{k_i-1} \to 1$ and hence, possibly after increasing K_4, that step 9 is reached in all iterations $k_i - 1$ with $k_i - 1 \geq K_4$.

As we have already seen, this leads to a contradiction. Hence, $\theta_\ell(w_{k_i}) + \mu_{k_i} \geq \epsilon$ for all i is not true. The proof is therefore completed since there exists a subsequence $\{k'_j\} \subset \{k_i\}$ for which $\lim_{j\to\infty} \theta(w_{k'_j}) = \lim_{j\to\infty} \theta_\ell(w_{k'_j}) + \mu_{k'_j} = 0$. □

The other situation is when the sequence $\{k_i\}$ of Lemma 4.9 contains a subsequence, where the iterates are added to the filter in step 8. As in the previous lemma we have the following result.

LEMMA 4.11. *Suppose that there exists an infinite sequence $\{k_i\}$ of iterations for which w_{k_i} is added to the filter in step 8 and, in addition,* $\lim_{i\to\infty} \theta(w_{k_i}) = 0$. *Then $\{k_i\}$ contains a subsequence $\{k'_j\}$ such that*

$$\lim_{j\to\infty} \theta(w_{k'_j}) = 0, \qquad \lim_{j\to\infty} \theta_\ell(w_{k'_j}) + \mu_{k'_j} = 0.$$

Proof. Let $\{k_i\}$ be a sequence of iterations such that w_{k_i} is added to the filter in step 8 and $\lim_{i\to\infty} \theta(w_{k_i}) = 0$. Suppose now that $\theta_\ell(w_{k_i}) + \mu_{k_i} \geq \epsilon > 0$ for all $k_i \geq K_0$ for some $K_0 \geq 0$. Then we have, by Lemma 2.6, that $\mu_{k_i} \geq a(\epsilon)$. By Lemma 2.5 and since w_{k_i} is added to the filter in step 8 we have

$$\begin{aligned} M_\mu \alpha^t_{k_i} a(\epsilon) &\leq m_{k_i}(w_{k_i}) - m_{k_i}(w_{k_i}(\Delta_{k_i})) + M_\theta \theta(w_{k_i}) \\ &< \kappa \theta(w_{k_i})^2 + M_\theta \theta(w_{k_i}). \end{aligned}$$

Thus, we obtain $\alpha^t_{k_i} \to 0$ and consequently $\Delta_{k_i} \to 0$. In particular, $\alpha^t_{k_i} \geq \Delta_{k_i}/M_s$ for large enough i, and since the restoration procedure is not called, we have $\theta(w_{k_i}) \leq \gamma_2 \Delta^{1+\beta}_{k_i}$ and conclude that

$$\begin{aligned} \Delta_{k_i} M_\mu \frac{a(\epsilon)}{M_s} &\leq m_{k_i}(w_{k_i}) - m_{k_i}(w_{k_i}(\Delta_{k_i})) + M_\theta \theta(w_{k_i}) \\ &< \kappa(\gamma_2 \Delta^{1+\beta}_{k_i})^2 + M_\theta \gamma_2 \Delta^{1+\beta}_{k_i}, \end{aligned}$$

which is a contradiction to $\Delta_{k_i} \to 0$. □

We put both situations together in the next lemma.

THEOREM 4.12. *Suppose that infinitely many iterates are added to the filter. Then there exists a subsequence $\{k_j\}$ such that*

$$\lim_{j\to\infty} \theta(w_{k_j}) = 0, \qquad \lim_{j\to\infty} \theta_\ell(w_{k_j}) + \mu_{k_j} = 0.$$

Proof. By Lemma 4.9 there exists a sequence $\{k_i\}$ of iterates such that w_{k_i} is added to the filter and $\lim_{i\to\infty} \theta(w_{k_i}) = 0$. As we have already observed there exists either a subsequence $\{k'_j\}$ of $\{k_i\}$ such that $w_{k'_j}$ are added to the filter before entering restoration or a subsequence $\{k'_j\}$ of $\{k_i\}$ such that $w_{k'_j}$ are added to the filter in step 8. In the first case the assertion follows from Lemma 4.10; in the second case from Lemma 4.11. □

Finally, we analyze the case where the algorithm runs infinitely but only finitely many iterates are added to the filter.

THEOREM 4.13. *Suppose that the algorithm runs infinitely and only finitely many iterates are added to the filter. Then*

$$\lim_{k\to\infty} \theta(w_k) = 0, \qquad \liminf_{k\to\infty} \theta_\ell(w_k) + \mu_k = 0.$$

Proof. The assumption says that for $k \geq K$, with K large enough, no further filter entry is added. Hence, the filter contains for all $k \geq K$ the same finitely many entries, and the restoration is never invoked. Thus, all new iterates $w_{k+1} \neq w_k$ are computed in step 10. We now show that step 10 is reached infinitely many times.

In fact, step 5 is reached in each iteration, and, by Lemma 4.6, step 7 is reached after finitely many reductions of Δ_k in step 11 (note that $\mu_k > 0$, since $x_k, z_k > 0$). Again, step 8 is reached after finitely many reductions of Δ_k. In fact, if $\theta(w_k) > 0$ then clearly

$$m_k(w_k) - m_k(w_k(\Delta_k)) < \kappa\theta(w_k)^2$$

for Δ_k sufficiently small and step 8 is reached. Otherwise, $\theta(w_k) = 0$ and $\theta_\ell(w_k) + \mu_k > 0$ and therefore $\rho_k \geq \eta$ for all Δ_k small enough (we may apply exactly the same arguments as at the end of the proof of Lemma 4.10). So, step 10 is always reached after finitely many reductions of Δ_k, producing always new iterates.

Since no further entry is added to the filter we know, cf. step 8, that in step 10 it always holds that

$$\theta_g(w_k) - \theta_g(w_{k+1}) \geq \eta(m_k(w_k) - m_k(w_k(\Delta_k))) \geq \eta\kappa\theta(w_k)^2.$$

Since this holds for all successful steps and $\{\theta_g(w_k)\}$ is bounded, we conclude that

$$\lim_{k\to\infty} \theta(w_k) = 0. \tag{4.15}$$

Now assume that $\theta_\ell(w_k) + \mu_k \geq \epsilon > 0$ for all $k \geq K$ and some $\epsilon > 0$. Then Lemma 2.6 yields again $\mu_k \geq a(\epsilon)$. Since the filter entries do not change for $k \geq K$, the test in step 5 is passed whenever $\Delta_k \leq \Delta_a(\epsilon)$ (cf. Lemma 4.6). Also, since $\theta_\ell(w_k) + \mu_k \geq \epsilon > 0$, we obtain as before that $\rho_k \geq \eta$ whenever $\Delta_k \leq \Delta'(a(\epsilon))$ for some $\Delta'(a(\epsilon)) > 0$. Finally, we know by Lemma 4.2.iii that for $\Delta_k^{in} \leq \Delta_{\min}(\epsilon)$ steps 2 and 3 yield $\Delta_k = \Delta_k^{in}$. Hence, we see that $\Delta_k \geq \delta(\epsilon) \stackrel{\text{def}}{=} \min\{\Delta_a(\epsilon)/2, \Delta'(a(\epsilon))/2, \nu\Delta_{\min}(\epsilon), \Delta_K\} > 0$ for $k \geq K$. Thus, step 10 is reached for all successful steps with $\Delta_k \geq \delta(\epsilon) > 0$ and we have, as above (using (4.15)), for k sufficiently large,

$$\begin{aligned}
\theta_g(w_k) - \theta_g(w_{k+1}) &\geq \eta(m_k(w_k) - m_k(w_k(\Delta_k))) \\
&\geq \eta M_\mu a(\epsilon)\alpha_k^t - \eta M_\theta \theta(w_k) \\
&\geq \eta M_\mu a(\epsilon) \min\left\{\frac{\delta}{M_s}, 1\right\} - \eta M_\theta \theta(w_k) \\
&\geq \eta M_\mu a(\epsilon) \min\left\{\frac{\delta}{M_s}, 1\right\},
\end{aligned}$$

where M_s is as before a uniform upper bound on $\max\{\|s_k^t\|, \|s_k^n\|\}$. This is a contradiction to the boundedness of $\theta_g(w_k)$ and the proof is complete. □

Our global convergence result (Theorem 4.1) can be obtained by combining Theorems 4.12 and 4.13.

5. The solver `ipfilter` for nonlinear programming. Our implementation of the interior-point filter method (Algorithm 3.1) is called `ipfilter`. The code is written in Fortran 90. In this section we review the main practical issues involved in `ipfilter`. For more details see (1.1).

5.1. Upper and lower bounds. The implemention of the primal-dual interior-point filter method (Algorithm 3.1) considered in `ipfilter` handles problems of the form

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } h(x) = 0, \\ &\qquad l \le x \le u, \end{aligned}$$

where $l \in (\{-\infty\} \cup \mathbb{R})^n$ and $u \in (\mathbb{R} \cup \{\infty\})^n$. Upper and lower bounds on the variables are thus considered explicitly avoiding in this case the introduction of slack variables. In some problems not all variables have upper and lower bounds, and `ipfilter` was adapted to take care of such situations, including the case where all the variables are free. The current version of the code also addresses unconstrained problems and problems with simple bounds but this is not treated in this paper.

Under this new formulation, the Lagrangian function becomes

$$\ell(w) = f(x) + h(x)^T y - (x - l)^T z^l - (u - x)^T z^u,$$

where $w = (x, y, z^l, z^u) \in \mathbb{R}^{3n+m}$ and $z^l \in \mathbb{R}^n$ and $z^u \in \mathbb{R}^n$ are the Lagrange multipliers associated with the lower and upper bounds, respectively. The quantity μ is now defined as

$$\mu = \frac{(x - l)^T z^l + (u - x)^T z^u}{2n},$$

and the centrality measure as

$$\theta_c(w) = \left\| \begin{pmatrix} (X - L) z^l - \mu e \\ (U - X) z^u - \mu e \end{pmatrix} \right\|,$$

where $L = \text{diag}(l)$ and $U = \text{diag}(u)$. Finally, the neighborhood $\mathcal{N}(\gamma, M, p)$ is now defined as

$$\begin{aligned} \mathcal{N}(\gamma, M, p) = \big\{ w \ : \ & l < x < u, \ (z^l, z^u) > 0, \ (X - L) z^l \ge \gamma \mu e, \\ & (U - X) z^u \ge \gamma \mu e, \ \theta_h(w) + \theta_\ell(w)^p \le M \mu \big\}. \end{aligned}$$

5.2. Linear systems. The systems of linear equations that define the normal and tangential steps are redefined as

$$\begin{pmatrix} H & A & -I & I \\ A^T & 0 & 0 & 0 \\ Z^l & 0 & X - L & 0 \\ -Z^u & 0 & 0 & U - X \end{pmatrix} \begin{pmatrix} \Delta x^n \\ \Delta y^n \\ \Delta z^{l,n} \\ \Delta z^{u,n} \end{pmatrix} = - \begin{pmatrix} 0 \\ h(x) \\ (X - L) z^l - \mu e \\ (U - X) z^u - \mu e \end{pmatrix} \tag{5.1}$$

and

$$\begin{pmatrix} H & A & -I & I \\ A^T & 0 & 0 & 0 \\ Z^l & 0 & X - L & 0 \\ -Z^u & 0 & 0 & U - X \end{pmatrix} \begin{pmatrix} \Delta x^t \\ \Delta y^t \\ \Delta z^{l,t} \\ \Delta z^{u,t} \end{pmatrix} = - \begin{pmatrix} \nabla_x \ell(w) \\ 0 \\ (1 - \sigma) \mu e \\ (1 - \sigma) \mu e \end{pmatrix}, \tag{5.2}$$

where H denotes $\nabla^2_{xx} \ell(w)$ or an approximation thereof, A denotes $\nabla h(x)$, $Z^l = \text{diag}(z^l)$, and $Z^u = \text{diag}(z^u)$. From now on, we will deal with both systems at the same time, considering a generic right-hand-side of the form (r_1, r_2, r_3, r_4).

Most of the computational effort of the algorithm is spent in solving these systems of linear equations. Instead of solving the nonsymmetric linear systems (5.1) and (5.2), we solve equivalent, smaller and symmetric systems, using some algebraic manipulations known in interior-point methods. We start by eliminating the last two block rows

$$\begin{aligned}\Delta z^l &= (X-L)^{-1}(r_3 - Z^l \Delta x),\\ \Delta z^u &= (U-X)^{-1}(r_4 + Z^u \Delta x),\end{aligned}$$

and writing

$$\begin{pmatrix} H + (X-L)^{-1}Z^l + (U-X)^{-1}Z^u & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} =$$

$$= \begin{pmatrix} r_1 + (X-L)^{-1}r_3 - (U-X)^{-1}r_4 \\ r_2 \end{pmatrix}.$$

To avoid the inversion of $X - L$ and $U - X$ in the matrix systems, we rewrite them as:

$$\begin{pmatrix} D^{1/2}HD^{1/2} + (U-X)Z^l + (X-L)Z^u & D^{1/2}A \\ (D^{1/2}A)^T & 0 \end{pmatrix} \begin{pmatrix} \overline{\Delta x} \\ \Delta y \end{pmatrix} = \begin{pmatrix} \bar{r}_1 + \bar{r}_3 - \bar{r}_4 \\ r_2 \end{pmatrix}, \tag{5.3}$$

where

$$\begin{aligned}D &= (X-L)(U-X),\\ \bar{r}_1 &= D^{1/2}r_1,\\ \bar{r}_3 &= (U-X)^{1/2}(X-L)^{-1/2}r_3,\\ \bar{r}_4 &= (U-X)^{-1/2}(X-L)^{1/2}r_4,\\ \Delta x &= D^{1/2}\overline{\Delta x}.\end{aligned}$$

Our implementation in `ipfilter` uses the sparse routines `MA27` from HSL [7] (the former Harwell Subroutine Library) to solve these symmetric systems. `MA27` computes a factorization $A = LDL^T$ of a symmetric matrix A, where L is a lower triangular matrix with ones in the diagonal and D is a block diagonal matrix formed by 1×1 or 2×2 diagonal blocks. We changed the following `MA27` parameters: as the threshold parameter for the numerical pivoting we choose 10^{-6} and as the pivoting tolerance we use 10^{-12}.

The parameter σ is chosen in the following way: If $\mu < 10^{-6}$ then $\sigma = \sigma_f^{\rm int} = 2.6 \times 10^{-3}$, otherwise $\sigma = \sigma^{\min} = 10^{-5}$.

We must also mention that we compute a safe step if the tangential step length $\alpha^t(\Delta)$ is less than a positive quantity ψ (in our implementation $\psi = 0.8$). In that case, we use $\sigma = \sigma_s^{\max} = 0.1$ (if $\alpha^t(\Delta) < 10^{-3}$) or $\sigma = \sigma_s^{\rm int} = 10^{-3}$ (otherwise) and recompute the tangential step s^t by resolving the system (5.2).

5.3. Control of inertia and conditioning. In `ipfilter` we introduce perturbations in the $(1,1)$ and $(2,2)$ blocks of the symmetric matrix M of the systems (5.3),

$$\begin{pmatrix} M_{11} + \epsilon_1 I & M_{12} \\ M_{12}^T & -\epsilon_2 I \end{pmatrix}, \tag{5.4}$$

where ϵ_1 and ϵ_2 are positive parameters. The $(1,1)$-block perturbation is related to the regularization often applied to make the approximation of the Hessian of the Lagrangian positive definite on the null space of the constraints. The $(2,2)$-block perturbation corresponds to the regularization of the equality constraints given by $h(x) - \epsilon_2 y = 0$.

The perturbations are chosen to force the inertia of M to be equal to $(n, m, 0)$ (n positive eigenvalues, m negative eigenvalues, and 0 zero eigenvalues), which is known to be related to a QP subproblem convexification. The inertia control scheme that we use (see Algorithm 5.1) is based on the one suggested by Wächter and Biegler [21].

ALGORITHM 5.1 (Inertia Control). Start with $\epsilon_1^{\text{last}} = 0$ at the beginning of the optimization.

In each iteration:

1. Attempt to factorize the system matrix (5.4), with $\epsilon_1 = \epsilon_2 = 0$. If the inertia is $(n, m, 0)$ or both $\theta_\ell(w)$ and $\theta_h(w)$ are small (less than 10^{-5}), solve the systems (5.3). Otherwise, go to step 2.
2. If the matrix is singular, set $\epsilon_2 = 10^{-20}$, otherwise, set $\epsilon_2 = 0$. Set also $\epsilon_1 = 0.5 \times 10^{-6}$ (if $\epsilon_1^{\text{last}} = 0$) or $\epsilon_1 = \max\{10^{-20}, \epsilon_1^{\text{last}}/3\}$ (otherwise).
3. Attempt to factorize the perturbed matrix (5.4). If the inertia is now $(n, m, 0)$ then set $\epsilon_1^{\text{last}} = \epsilon_1$ and solve the systems (5.3). Otherwise, go to step 4.
4. If $\epsilon_1^{\text{last}} = 0$, set $\epsilon_1 = 10 \times \epsilon_1$, otherwise $\epsilon_1 = 2 \times \epsilon_1$.
5. If $\epsilon_1 > 10^6$, set $\epsilon_1^{\text{last}} = \epsilon_1$, $\epsilon_1 = 0$, and $\epsilon_2 = 10^{-20}$ and solve the systems (5.3). Otherwise, go back to step 3.

Note that an approximation to the inertia of a matrix is readily available from the application of `MA27` [7].

5.4. Initial point and warm start. Given the initial primal point x_{in} associated with each problem, `ipfilter` first projects this point onto the interior of the box defined by the bound constraints and then, starting from this point, it applies a number (5 in the current implementation) of iterations of Newton's method (subject to the bound constraints) to minimize the function

$$f(x) + \rho\, h(x)^T h(x) - \mu_{in} \sum_{i=1}^{n} \log(x_i - l_i) - \mu_{in} \sum_{i=1}^{n} \log(u_i - x_i),$$

with $\mu_{in} = 10^3$ and ρ chosen in the following way: If $\|h(x_{in})\|_1 > 50$ then $\rho = \rho^{\max} = 10^5$; otherwise if $\|h(x_{in})\|_1 < 10^{-10}$ then $\rho = \rho^{\min} = 1$, otherwise $\rho = \rho^{\text{int}} = 1.77 \times 10^3$. The resulting point is the initial iterate x_0. The iterations taken in this initial procedure are counted in the tables.

The initial multipliers with respect to the equality constraints are first set to:

$$y_0 = \operatorname{argmin} \|\nabla f(x_0) + \nabla h(x_0)\, y - z_{in}^l + z_{in}^u\|,$$

where, componentwise, we have $z_{in}^l = \mu_{in}/(x_0 - l)$ and $z_{in}^u = \mu_{in}/(u - x_0)$. If y_0 obtained in this way is too large, i.e., if $\|y_0\|_\infty > y_{\max}$ (with $y_{\max} = 2 \times 10^3$ in our implementation), then this initial value for y_0 is discarded and the algorithm starts with $y_0 = 0$. Then, the initial multipliers with respect to the bounds are set to:

$$\begin{aligned} z_0^l &= \max\{\nabla f(x_0) + \nabla h(x_0) y_0 + z_{in}^u, z_{in}^l\}, \\ z_0^u &= \max\{-\nabla f(x_0) - \nabla h(x_0) y_0 + z_0^l, z_{in}^u\}. \end{aligned}$$

The linear systems resulting from the application of Newton's method in the primal warm start procedure and from the calculation of y_0 are converted in a 2×2 symmetric block format by introducing an auxiliar, later discarded vector of variables. The sparse routines `MA27` from HSL [7] are also used to solve these symmetric systems. Our implementation uses the routines `aplb` and `amub` from the SPARSKIT Library (Version 2) [16] to perform the matrix multiplications needed in the primal warm start procedure.

5.5. Restoration. The purpose of a restoration algorithm in the context of Algorithm 3.1 is to find a point $w_{k+1} \in \mathcal{N}(\gamma, M, p)$ acceptable to the filter and such that the condition $\theta(w_{k+1}) \leq \Delta_{k+1}^{in} \min\{\gamma_1, \gamma_2(\Delta_{k+1}^{in})^\beta\}$ is satisfied with $\Delta_{k+1}^{in} = \Delta_k$. The implementation chosen for `ipfilter` follows the restoration algorithm proposed in [18, Algorithm 2] where essentially our primal-dual step computation is applied to minimize the value of

$$\theta_2(w) \stackrel{\text{def}}{=} \frac{1}{2}\left(\theta_h(w)^2 + \theta_c(w)^2\right) = \frac{1}{2}\left(\|h(x)\|^2 + \|Xz - \mu e\|^2\right).$$

The iterates of this restoration algorithm are forced to stay in the central neighborhood $\mathcal{N}(\gamma, M, p)$ and to satisfy a sufficient decrease condition for θ_2. We have tested also the alternative restoration scheme suggested in [18] but its performance was not superior.

We have introduced some practical modifications to [18, Algorithm 2] to put emphasis on the reduction of θ_2 and therefore improve its efficiency. Thus, we replaced H by the identity matrix in the primal-dual systems defining the normal and tangential components. Also, in the right-hand-side vector of the linear system defining the tangential step we have replaced the term $-\nabla_x \ell(w)$ by zero. In the restoration phase, we always introduce a perturbation in the $(2,2)$ block of the system matrix, of the form $-\epsilon_2 I$, where $\epsilon_2 = \min\{\mu_k, 10^{-8}\}$.

5.6. Other parameters and stopping criterion. The γ and M parameters in $\mathcal{N}(\gamma, M, p)$ are computed as follows:

$$\gamma = \min\left\{10^{-3}, \frac{1}{2\,\mu_0} \min_{i=1,\ldots,n} \left\{\min\left\{\left((X_0 - L)z_0^l\right)_i, \left((U - X_0)z_0^u\right)_i\right\}\right\}\right\}$$

and

$$M = \max\left\{10^3, \frac{10^3}{\mu_0}\left(\theta_h(w_0) + \theta_\ell(w_0)\right)\right\}.$$

The current implementation sets $p = 1$ in $\mathcal{N}(\gamma, M, p)$ since it supports only the use of second-order derivatives. The update of τ_k obeys to

$$\tau_k \ = \ 1 - \min\{10^{-2}, 10^{-2}\mu_k^2\}.$$

For the initial step length parameter Δ_0^{in} we have chosen the value 10^5. The update of Δ_k^{in} in step 9 took the form: $\Delta_{k+1}^{in} = 2\Delta_k$ when $\rho_k > \eta_2$ and $\Delta_{k+1}^{in} = \Delta_k$ otherwise. We have picked $\eta = \eta_1 = 10^{-4}$, $\eta_2 = 0.8$, and $\nu = 0.5$. For the remaining parameters of the main algorithm we have chosen:

$$\gamma_1 = 0.5,\ \gamma_2 = 1,\ \beta = 0.75,\ \kappa = 0.1,\ \text{and}\ \gamma_{\mathcal{F}} = 10^{-3}.$$

The parameters needed to update Δ_k in the restoration procedure [18, Algorithm 2] were set similarly as in the main algorithm. The value of Δ_k in the restoration is kept constant when the ratio of actual over predicted decreases for θ_2 was between ξ_1 and ξ_2 and doubled when this ratio is larger then ξ_2. We chose $\xi_1 = 10^{-5}$ and $\xi_2 = 0.5$. The parameter ν to enforce the iterates to lie in the central neighborhood was set to 0.5. Finally we set $\sigma = 1$.

The stopping criterion used by `ipfilter` is as similar to the one of `ipopt` [21] as possible. In fact, `ipfilter` is stopped if

$$\max\left\{\|h(x_k)\|_\infty, \frac{\left\|\begin{pmatrix} (X_k - L)z_k^l \\ (U - X_k)z_k^u \end{pmatrix}\right\|_\infty}{s_c}, \frac{\|\nabla_x \ell(w_k)\|_\infty}{s_\ell}\right\} < 10^{-8},$$

	# problems	
dimensions	an old CUTE set	CUTEr (Sept. 2008)
$n < 1000$	388 (45)	390 (56)
$1000 \leq n < 10000$	76 (1544)	182 (3194)
$n \geq 10000$	5 (7979)	59 (6354)
total	469	631

TABLE 6.1
Dimensions of the test problems (n is the number of variables and in brackets it is indicated the average number of equality constraints).

	# problems	
problem class	an old CUTE set	CUTEr (Sept. 2008)
equality constrained	245	327
inequality constrained	177	226
mixed (equalities and inequalities)	47	78
linearly constrained	171	205
nonlinearly constrained	298	426
quadratic programming	91	103

TABLE 6.2
A summary of the test problems.

where s_c and s_ℓ are scaling factors given by

$$s_c = 10^{-2} \max \left\{ 100, \frac{\|z_k^l\|_1 + \|z_k^u\|_1}{2n} \right\},$$
$$s_\ell = 10^{-2} \max \left\{ 100, \frac{\|y_k\|_1 + \|z_k^l\|_1 + \|z_k^u\|_1}{m + 2n} \right\},$$

or if

$$\Delta_k < 10^{-10},$$

or if the number of iterations reaches 1000.

6. Numerical results. We report the results of `ipfilter` on a set of problems from the CUTEr test set [14] with at least one constraint (not of the simple bound type) for which $n \geq m$ in our formulation. The problem dimensions are summarized in Table 6.1. In Table 6.2, we give some information on the different types of problems of this test set[1]. Since we tested `ipfilter` mostly on an old CUTE set (also reported in these tables), we also present the results for this test set. The two lists of problems are given in (1.1).

We compared the performance of `ipfilter` (version 0.2) with the interior-point barrier filter solver `ipopt`, version 3.5.1 in C++, developed by Wächter and Biegler [21]. The tests were run on a Fujitsu-Siemens Celsius V810 workstation (4 GB RAM, 2 processors AMD 2.2GHz). The results are summarized in Table 6.3. We declared failure for both codes when the step size becomes too small (either in the main algorithm or in the restoration), the maximum number of (primal-dual) iterations is reached or a budget of 500 minutes of CPU time is spent, before the termination criterion is met for a stopping tolerance of 10^{-8}.

[1]We excluded 91 problems which required an increase of the default size parameters of `sifdec` in CUTEr.

	an old CUTE set		CUTEr (Sept. 2008)	
	`ipfilter`	`ipopt`	`ipfilter`	`ipopt`
# problems solved	449	448	532	549
% robustness	95.74%	95.52%	84.34%	87.00%
# average iterations	27.55	27.19	47.44	38.58
# problems solved (< 500 iter.)	449	447	525	545
% robustness (< 500 iter.)	95.74%	95.31%	83.20%	86.37%
# average iterations (< 500 iter.)	27.55	25.78	37.51	34.14

TABLE 6.3
Robustness and average number of iterations.

	an old CUTE set		CUTEr (Sept. 2008)	
	`ipfilter`	`ipopt`	`ipfilter`	`ipopt`
# problems solved	91	88	97	93
% robustness	100.00%	96.70%	94.17%	90.29%
# average iterations	26.74	36.45	42.35	47.09
# problems solved (< 500 iter.)	91	88	96	92
% robustness (< 500 iter.)	100.00%	96.70%	93.20%	89.32%
# average iterations (< 500 iter.)	26.74	36.45	33.90	40.96

TABLE 6.4
Robustness and average number of iterations (quadratic programming problems).

We made several other tests which are not reported in detail in this paper. For instance, we tested the impact on the performance of `ipfilter`, of the new optimality filter entries, (2.11) and (2.12), compared to the one suggested in [18] and given in (2.10). Regardless of the measure chosen (robustness, number of iterations, or final objective function value), the optimality filter entry (2.12) seems to be the best among the three.

We also analyzed the performance of the two codes on the different problem classes of Table 6.2. The relative performances for the different classes followed the general pattern, except for quadratic programming where `ipfilter` seems to be currently slightly better than `ipopt` (see Table 6.4).

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Appendix: Proof of Lemma 2.2.

For the proof of Lemma 2.2 we use the following estimates.

LEMMA 6.1. *The following estimates hold, using the conventions* $0^0 = 1$, $0^\alpha = 0$, $\alpha > 0$.

$$\forall\, a, b \geq 0,\ \alpha \in [0,1]: \quad (a+b)^\alpha \leq a^\alpha + b^\alpha. \tag{6.1}$$

$$\forall\, \alpha, t \in [0,1]: \quad t^\alpha + (1-t)^\alpha \leq 2^{1-\alpha}. \tag{6.2}$$

Proof. Proof of (6.1): The cases $a = 0$ or $b = 0$ or $\alpha = 0$ are obvious. We thus can assume $a, \alpha > 0$ and bracket out a. Then we have to prove

$$f(t) \stackrel{\text{def}}{=} 1 + t^\alpha - (1+t)^\alpha \geq 0 \quad \forall\, 0 \leq t \leq 1.$$

We have

$$f(0) = 0, \quad f'(t) = \alpha t^{\alpha-1} - \alpha(1+t)^{\alpha-1} \geq 0 \ \ (t > 0).$$

Proof of (6.2): Consider first the case $\alpha = 0$. Then

$$t^0 + (1-t)^0 = 1 + 1 = 2^{1-0}.$$

For the case $\alpha \in (0,1]$ and $t \in \{0,1\}$ we obtain

$$t^\alpha + (1-t)^\alpha = 0^\alpha + 1^\alpha = 1 \le 2^{1-\alpha}.$$

Finally, let $\alpha \in (0,1]$ and $t \in (0,1)$ and consider

$$f(t) \stackrel{\text{def}}{=} t^\alpha + (1-t)^\alpha.$$

Then

$$f'(t) = \alpha t^{\alpha-1} - \alpha(1-t)^{\alpha-1} \begin{cases} \ge 0 & 0 < t < 1/2, \\ = 0 & t = 1/2, \\ \le 0 & 1/2 < t < 1. \end{cases}$$

Therefore,

$$f(t) \le f(1/2) = 2 \cdot 2^{-\alpha} = 2^{1-\alpha}.$$

□

We are now able to prove Lemma 2.2.

Proof. (of Lemma 2.2) We have

$$\nabla g(x) = p\|x\|^{p-2}x \;\; (x \ne 0), \quad \nabla g(0) = 0 \;\; (p > 1).$$

Furthermore,

$$\nabla^2 g(x) = p\|x\|^{p-2}I + p(p-2)\|x\|^{p-4}xx^T \;\; (x \ne 0), \quad \nabla^2 g(0) = 2I \;\; (p = 2).$$

Let $x, y \in \mathbb{R}^n$ be arbitrary and let τ be the minimizer of $t \in [0,1] \mapsto \|(1-t)x + ty\|$. Setting $z = (1-\tau)x + \tau y$ we have $z \perp (y-x)$.

We first consider the situation $\rho \stackrel{\text{def}}{=} \|z\| > 0$ and prove the second part of the lemma. The eigenvalues of $\nabla^2 g(x)$ are:

$$p\|x\|^{p-2} \;\; (\text{multiplicity } n-1), \quad p\|x\|^{p-2} + p(p-2)\|x\|^{p-2} = p(p-1)\|x\|^{p-2}.$$

Hence, since $p \le 2$

$$\|\nabla^2 g(x)\| = p\|x\|^{p-2}.$$

Therefore, we can estimate

$$\|\nabla g(y) - \nabla g(x)\| \le \int_0^1 \|\nabla^2 g(x + t(y-x))(y-x)\|\, dt \le p\rho^{p-2}\|y-x\|.$$

Now we turn to the first assertion of the lemma. We distinguish two cases.

If $\rho \stackrel{\text{def}}{=} \|z\| \le \|y-x\|/2$ we obtain by using (6.1) and (6.2)

$$\begin{aligned}
&\|\nabla g(y) - \nabla g(x)\| \le \|\nabla g(y)\| + \|\nabla g(x)\| = p\|y\|^{p-1} + p\|x\|^{p-1} \\
&= p(\|z\|^2 + (1-\tau)^2\|y-x\|^2)^{\frac{p-1}{2}} + p(\|z\|^2 + \tau^2\|y-x\|^2)^{\frac{p-1}{2}} \\
&\le 2p\|z\|^{p-1} + p(\tau^{p-1} + (1-\tau)^{p-1})\|y-x\|^{p-1} \\
&\le 2p\|z\|^{p-1} + 2^{2-p}p\|y-x\|^{p-1} \\
&\le 2 \cdot 2^{1-p}p\|y-x\|^{p-1} + 2^{2-p}p\|y-x\|^{p-1} \\
&= 2^{3-p}p\|y-x\|^{p-1}.
\end{aligned}$$

If $\rho \stackrel{\text{def}}{=} \|z\| > \|y-x\|/2$ we can use the assertion already proved to arrive at

$$\|\nabla g(y) - \nabla g(x)\| \le p\rho^{p-2}\|y-x\| \le 2^{2-p}p\|y-x\|^{p-1}.$$

□

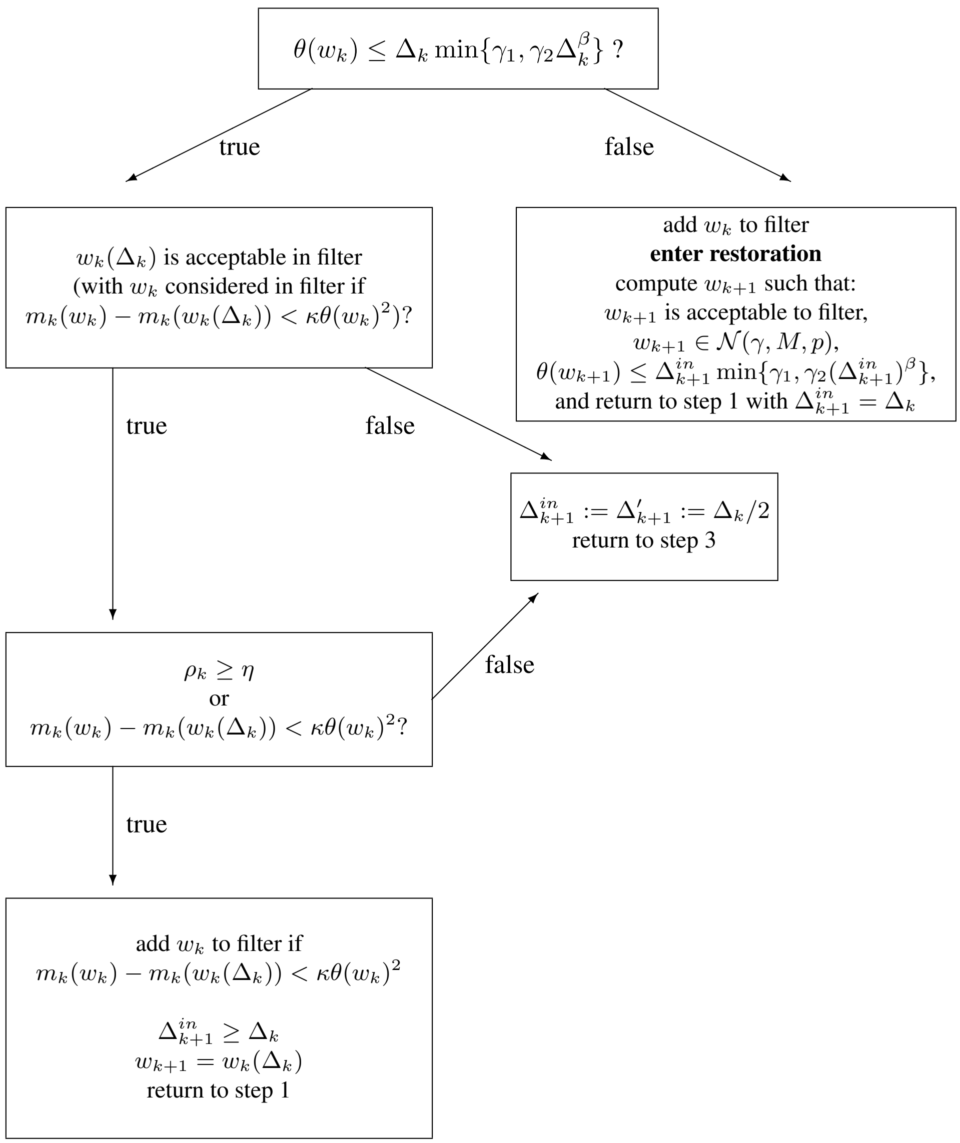

FIG. 3.1. *Steps 4-11 of Algorithm 3.1.*