PRIMAL-DUAL INTERIOR-POINT METHODS FOR PDE-CONSTRAINED OPTIMIZATION

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Abstract. This paper provides a detailed analysis of a primal-dual interior-point method for PDE-constrained optimization. Considered are optimal control problems with control constraints in L^p. It is shown that the developed primal-dual interior-point method converges globally and locally superlinearly. Not only the easier L^∞-setting is analyzed, but also a more involved L^q-analysis, $q < \infty$, is presented. In L^∞, the set of feasible controls contains interior points and the Fréchet differentiability of the perturbed optimality system can be shown. In the L^q-setting, which is highly relevant for PDE-constrained optimization, these nice properties are no longer available. Nevertheless, a convergence analysis is developed using refined techniques. In particular, two-norm techniques and a smoothing step are required. The L^q-analysis with smoothing step yields global linear and local superlinear convergence, whereas the L^∞-analysis without smoothing step yields only global linear convergence.

1. Introduction.

This paper is concerned with the analysis of primal-dual interior-point methods for optimization problems with PDE- and pointwise inequality constraints. We assume that the problem has optimal control structure and that the inequality constraints are posed on the controls only. In contrast to state constraints, this situation allows for a rigorous analysis. Related investigations of other Newton-based algorithms were conducted in, e.g., [4, 8, 12, 14, 16, 22, 20, 21, 23, 25, 26] for comparable problem settings. For primal-dual interior-point methods, although intensively investigated in finite dimensional mathematical programming, see, e.g., [7] and the references therein, only little rigorous theory is available in the function space framework of optimal control problems. Earlier investigations of modern optimization methods in function space have resulted in valuable deep understanding of algorithms for PDE constrained optimization. In particular, in all analyses, a certain problem structure is required for a successful local convergence analysis. A common theme is that an L^p-setting for the inequalities is required and that a smoothing property or smoothing step must be available. Furthermore, the usual backtracking in interior-point methods to keep iterates strictly positive has to be augmented by suitable projection techniques, at least if the primal-dual Newton step for the control is not in L^∞. Finally, integrated barriers are the appropriate choice, which result in a weighting of the pointwise barriers after discretization. All of these crucial ingredients are not visible in the finite dimensional analysis. A further important benefit of an abstract analysis in function space is that it is the prerequisite for proving mesh independence results, see, e.g., [1, 2, 9].

The purpose of this paper is to give a rigorous analysis of the global and fast local convergence of a primal-dual interior-point method for PDE-constrained optimization. The analysis covers not only the (easier) L^∞-setting but also the quite involved but in practice highly relevant L^q-setting, $q < \infty$. The crucial point is that for the analysis in the L^∞-setting one needs that the corresponding adjoint state (i.e., the Lagrange multiplier for the state equation) is also in L^∞, which is usually not the case for complex systems like, e.g., the Navier-Stokes equations [6, 10, 19]. One of the difficulties in the L^q-setting, $q < \infty$, is that the set of feasible controls does not contain interior points with respect to the L^q-topology. As a consequence, the barrier function is not Fréchet-differentiable in L^q. Moreover, the Newton steps for the control and for the multipliers of the control constraints live in general only in L^q and therefore we have to work in an L^q-setting for the control and the multipliers even if the control is confined by L^∞-bounds. This requires elaborate techniques, including a suitable scaling of the primal-dual Newton system, a two-norm approach, and a smoothing step.

In the L^q-setting the central path can touch the boundary of the set of feasible controls on a set of measure zero. We will see that nevertheless interior-point methods make sense in connection

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with a projection onto an appropriate neighborhood of the central path. We empasize that also in the context of state constraints the central path may touch the boundary of the feasible set. The results and proof techniques of the present paper show that this does not necessarily lead to a failure of interior-point methods and may be valuable also in the context of state constraints. The use of two-norm technique and the introduction of a smoothing step allows not only to go beyond the L^∞-setting. It also results in a globally linearly and locally superlinearly convergent primal-dual interior-point algorithm, whereas the L^∞-setting without smoothing step results only in a globally linearly convergent method.

We now give a short overview of the available literature on interior-point methods in function space. In the context of optimal control with ODEs and control constraints, [24] investigates the properties of the central path in an L^∞-setting and proposes a corresponding continuation method. For a similar setup, a linearly convergent primal-dual interior-point short step method was investigated in [23] in an L^∞-framework. Furthermore, a primal interior-point method for linear-quadratic optimal control problems with an elliptic PDE and control constraints was investigated in [25]. In this approach, the control is eliminated from the perturbed optimality system and linear convergence of a short step path following method is proved. Superlinear convergence of this method was proved in [16] under H^2-regularity assumptions. The convergence of a primal-dual interior-point method for linear-quadratic elliptic control problems with mixed control-state constraints was investigated in [14]. The problem can be reformulated as a control problem with control constraints having special structure. The analysis of the central path and linear convergence of a short step path following method is carried out in an L^∞-framework. In addition to the mentioned references, numerical evaluations of interior-point methods in the context of PDE constrained optimization have been conducted in, e.g., [3, 13].

The paper is organized as follows: In section 2 the considered problem class is described and it is illustrated that elliptic optimal control problems fit into this class. Then, first order optimality (KKT) conditions are derived. As a first step towards interior-point methods, a barrier problem is formulated, its unique solvability is proved, and optimality conditions are stated that result in perturbed KKT conditions that form the basis for the primal dual Newton step. Section 3 presents and illustrates a functional analytic setting that is used in the rest of the paper. In section 4, properties of the central path are derived, in particular the boundedness of the dual variables in L^q and the boundedness of the central path. Section 5 is devoted to the analysis of the primal-dual Newton system on a suitable neighborhood of the central path. A key result is the uniformly bounded invertibility of the suitably scaled linear operator in the primal dual Newton system on bounded subsets of the neighborhood. As a simple consequence, the norm of the inverse of the unscaled operator is uniformly bounded by $O(1/\sqrt{\mu})$ on bounded subsets of the neighborhood. The Hölder continuity of the central path is proved in section 6. The conceptual primal-dual interior-point method is formulated in section 7. It includes a projection onto the neighborhood of the central path that replaces the usual backtracking. In section 8, the method is analyzed in the L^∞-setting. Quadratic local convergence towards the central path and global linear convergence are proved. Finally, in section 9, the more involved analysis of the method in L^q, $q < \infty$, is carried out. As for other approaches, an inevitable norm gap occurs that has to be closed by a smoothing step. Such a smoothing step is derived and incorporated in the algorithm. For the resulting method, global linear and local superlinear convergence is proved. Section 10 presents numerical results.

Notations. We denote the L^p-norm by $\|\cdot\|_p$, $1 \le p \le \infty$. For Banach spaces X, Y we denote by $\mathcal{L}(X,Y)$ the space of bounded linear operators from X to Y equipped with the operator norm $\|\cdot\|_{X,Y}$. X^* is the dual space of a Banach space X and $\langle\cdot,\cdot\rangle_{X^*,X}$ is the corresponding dual pairing. By $\mathrm{leb}\,(\cdot)$ we denote the Lebesgue measure on $\mathbb{R}^n$. Throughout the paper equalities and inequalities between L^p-functions are meant pointwise almost everywhere. If $X \subset Y$ is a continuous embedding, we write $I_{X,Y}$, $I_{X,Y}x = x$ for the embedding operator. Sometimes, if no confusion is to be expected, we save space by writing I instead of $I_{X,Y}$. If $F : X \to Y$ is an operator between

Banach spaces, we denote by F' its first Fréchet derivative and by F'' its second Fréchet derivative, if they exist. If X is a product space, then $x = (x_1, \dots, x_m) \in X$ has $m \ge 2$ components and we denote by F_{x_i} the partial Fréchet-derivative with respect to x_i, by $F_{x_i x_j}$ the derivative of F_{x_i} with respect to x_j, etc., if they exist.
Occasionally, given Banach spaces $X_1 \subset X_0$, $Y_1 \subset Y_0$, where "$\subset$" means continuous embedding, we will use the notation $\mathcal{L}(X_0, Y_1) \subset \mathcal{L}(X_1, Y_0)$ to indicate that every operator $A \in \mathcal{L}(X_0, Y_1)$ induces an operator $\tilde{A} \in \mathcal{L}(X_1, Y_0)$ via $\tilde{A}x := Ax$, $x \in X_1$.

2. Control constrained optimal control problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with sufficiently smooth boundary. We consider the optimal control problem with control constraints

$$\min_{y \in Y, u \in U} J(y,u) \quad \text{s.t.} \quad c(y,u) = 0, \quad a \le u \le b, \tag{2.1}$$

where $U = L^p(\Omega)$, $p \in [2, \infty)$, $a, b \in L^\infty$, $b - a \ge \nu > 0$, and Y is a Banach space. We set

$$\mathcal{B} := \{u \in U \ : \ a \le u \le b\}, \quad x = (y,u), \quad X := Y \times U,$$

and assume that there exists an open set $\mathcal{D} \subset L^p(\Omega)$, $\mathcal{D} \supset \mathcal{B}$ such that

(A1) $J : Y \times U \to \mathbb{R}$, $c : Y \times U \to \Lambda$ are twice locally Lipschitz-continuously differentiable, Λ is a Banach space, and there exist uniform Lipschitz constants on bounded subsets of $Y \times \mathcal{B}$.

(A2) $c_y(y,u) \in \mathcal{L}(Y, \Lambda)$ has a bounded inverse for all $(y,u) \in Y \times \mathcal{D}$ and $\|c_y(y,u)^{-1}\|_{\Lambda, Y}$ is uniformly bounded on bounded subsets of $Y \times \mathcal{B}$.

(A3) For all $u \in \mathcal{D}$ there exists a unique solution $y = y(u) \in Y$ of

$$c(y(u), u) = 0$$

and there exists $M_y > 0$ with

$$\|y(u)\|_Y \le M_y \quad \forall\, u \in \mathcal{B}.$$

(A4) The reduced objective functional

$$u \in (\mathcal{B}, \|\cdot\|_\infty) \mapsto J(y(u), u) =: \hat{J}(u)$$

is lower semicontinuous w.r.t. sequential L^∞-weak* convergence.

REMARK 2.1. By the implicit function theorem (A1)–(A3) ensure that $u \in \mathcal{D} \mapsto y(u) \in Y$ and $u \in \mathcal{D} \mapsto J(y(u), u)$ are twice locally Lipschitz-continuously differentiable and in addition Lipschitz continuous on $\mathcal{B}$.

For convenience we identify $U^* = L^p(\Omega)^*$ with $L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, via the dual pairing

$$\langle v, u \rangle_{U^*, U} = \langle v, u \rangle := \int_\Omega vu \, d\xi.$$

We recall that in function spaces of distributions it is common practice to extend $\langle \cdot, \cdot \rangle$ to the distributional dual pairing. In our examples, we typically work with the Sobolev spaces $H_0^1(\Omega)$ and $\tilde{Y} = H_0^1(\Omega) \cap H^2(\Omega)$. The dual spaces with respect to the dual pairing $\langle \cdot, \cdot \rangle$ then result in the following continuous and dense embeddings:

$$\tilde{Y} \subset H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) = H_0^1(\Omega)^* \subset \tilde{Y}^*.$$

In this sense, we can (and will) interpret H_0^1-functions as (nice) L^2-functions and L^2-functions as (nice) H^{-1}-functions (the latter are generalized functions). Furthermore, we will omit operators of the form $I_{Y_1,Y_2} : y \in Y_1 \mapsto y \in Y_2$ if $Y_1 \subset Y_2$ is continuously embedded. In this case, we also have $I^*_{Y_1,Y_2} : y' \in Y_2^* \mapsto y' \in Y_1^*$, thus $I^*_{Y_1,Y_2} = I_{Y_2^*,Y_1^*}$, i.e., the adjoint acts like the identity.

PROPOSITION 2.2. *Under assumptions* (A1)–(A4) *problem (2.1) has a solution.*

Proof. Take a minimizing sequence $(y(u_k), u_k)$. Since $(u_k) \subset \mathcal{B}$, it is bounded in L^∞ and has a weak*-convergent subsequence, which we denote again by (u_k) for simplicity, with limit $\bar{u} \in \mathcal{B}$. But by (A4) we have

$$\limsup_{k\to\infty} J(y(u_k), u_k) \geq J(y(\bar{u}), \bar{u})$$

and thus $(y(\bar{u}), \bar{u})$ solves (2.1), since $(y(u_k), u_k)$ is a minimizing sequence. ∎

2.1. An example. As a standard example we consider the following elliptic control problem

$$\begin{aligned} \min_{y,u} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2} + \frac{\alpha}{2}\|u\|^2_{L^2} \\ \text{s.t.} \quad & -\Delta y = u \quad \text{in } \Omega, \\ & y = 0 \quad \text{in } \partial\Omega, \\ & a \leq u \leq b \quad \text{in } \Omega. \end{aligned} \tag{2.2}$$

Here, $\Omega \subset \mathbb{R}^n$ and $a, b \in L^\infty(\Omega)$ are as specified above, $u \in U := L^2(\Omega)$ is the control, $y \in H^1(\Omega)$ is the state, $y_d \in L^2(\Omega)$ is the desired state, and $\alpha > 0$ is a regularization parameter.

There are at least two reasonable ways to choose the functional analytic setting. We choose $p = 2$, $\mathcal{D} = U$ and have $U^* = U = L^2(\Omega)$.

2.1.1. First setting. The first setting is to consider the usual weak solution of the state equation. Here, the state space is $Y = H^1_0(\Omega)$ and the PDE is considered in the weak form

$$\int_\Omega \nabla y(\xi)^T \nabla v(\xi)\, d\xi = \int_\Omega u(\xi)v(\xi)\, d\xi \quad \forall\, v \in H^1_0(\Omega).$$

This results in the abstract state equation

$$c(y, u) := Ay - u = 0 \quad \text{in } \Lambda$$

with $\Lambda = Y^* = H^{-1}(\Omega)$. Note that, as mentioned earlier, we have omitted the embedding operator $I_{L^2,H^{-1}} \in \mathcal{L}(L^2(\Omega), \Lambda)$, $I_{L^2,H^{-1}}u = u$. The operator $A \in \mathcal{L}(Y, \Lambda)$ is defined by

$$\langle Ay, v\rangle_{H^{-1},H^1_0} = \langle Ay, v\rangle = \int_\Omega \nabla y(\xi)^T \nabla v(\xi)\, d\xi \qquad \forall\, y, v \in H^1_0(\Omega).$$

It is well known that $A \in \mathcal{L}(Y, \Lambda)$ is invertible. Clearly,

$$J(y, u) = \frac{1}{2}\|y - y_d\|^2_2 + \frac{\alpha}{2}\|u\|^2_2$$

is twice continuously differentiable with

$$\begin{aligned} &\langle J_y(y, u), v\rangle_{Y^*,Y} = \langle y - y_d, v\rangle = (y - y_d, v)_2, \\ &\langle J_u(y, u), w\rangle_{U,U^*} = \alpha\langle u, w\rangle = \alpha(u, w)_2, \\ &\langle J_{yy}(y, u)v_2, v_1\rangle_{Y^*,Y} = (v_1, v_2)_2, \\ &\langle J_{uu}(y, u)w_2, w_1\rangle_{U^*,U} = \alpha(w_1, w_2)_2. \end{aligned}$$

In short notation:

$$J_y(y, u) = y - y_d, \quad J_u(y, u) = \alpha u, \quad J_{yy}(y, u) = I_{Y^*,Y}, \quad J_{uu}(y, u) = \alpha I_U.$$

Furthermore, $c(y,u) = Ay - u$ is twice continuously differentiable with

$$c_y(y,u) = A, \quad c_u(y,u) = -I_{L^2,H^{-1}}, \quad c''(y,u) = 0.$$

The uniform Lipschitz constants on bounded sets for c, J' and their derivatives are clear due to bounded linearity. The uniform Lipschitz continuity of J on bounded sets follows from the boundedness of J' on bounded sets. Hence, (A1) is shown. (A2) follows since

$$c_y(y,u) = A$$

is constant and $A \in \mathcal{L}(Y, \Lambda)$ is invertible. (A3) follows from $y(u) = A^{-1}Bu$. Finally (A4) is satisfied since $u \in L^2 \mapsto J(A^{-1}u, u) \in \mathbb{R}$ is convex and continuous, hence sequentially weakly lower semicontinuous. As a consequence, $\hat{J}$ is also lower semicontinuous w.r.t. sequential L^∞-weak* convergence.

2.1.2. Second setting. For the analysis to follow, it will be useful to exploit additional regularity of the state and the adjoint state. This motivates to consider the following setting for (2.2):

From the assumptions on Ω and standard regularity results for elliptic equations it follows that the solution of the state equation enjoys more regularity, namely, $y \in \tilde{Y} := H_0^1 \cap H^2$ (we use a " ˜ " to distinguish from the first setting). Hence, we can write the state equation also as follows:

$$\tilde{c}(y,u) := \tilde{A}y - u = 0 \quad \text{in } \tilde{\Lambda} := L^2$$

with $\tilde{A} \in \mathcal{L}(\tilde{Y}, \tilde{\Lambda})$. Again $\tilde{A}$ is invertible in $\mathcal{L}(\tilde{Y}, \tilde{\Lambda})$.

Since the Laplace operator is symmetric, we obtain also H^2-regularity for the adjoint state. This will be important, since in assumption (A5)$_q$, we will in particular require that there exists $q \in (p, \infty]$ such that the adjoint state λ satisfies

$$c_u^*(y,u)\lambda \in L^q(\Omega), \quad \text{i.e., for (2.2),} \quad -\lambda \in L^q(\Omega).$$

The larger q can be chosen, the better. In the first setting, we would be limited to q-values for which the embedding $H_0^1 \subset L^q$ holds. If we want to increase q, (e.g., $q = \infty$ for $n = 3$), it is advantageous to exploit any regularity that is available for λ, and thus an $H_0^1 \cap H^2$-setting is beneficial (if it is applicable).

Just as before, we can calculate derivatives and verify the assumptions (A1)–(A4).

2.2. Optimality conditions. If we define the closed convex set

$$\mathcal{C} := \{0\} \times \mathcal{B} \subset \Lambda \times U$$

and the constraint function

$$h(y,u) := \begin{pmatrix} c(y,u) \\ u \end{pmatrix}$$

then the constraint in (2.1) can be written as

$$h(y,u) \in \mathcal{C}.$$

Denote for $(\lambda, z) \in \Lambda^* \times U^*$ the Lagrangian function for the abstract problem

$$\min_{y \in Y, u \in U} J(y,u) \quad \text{s.t.} \quad h(y,u) \in \mathcal{C}$$

by

$$L(y,u,\lambda,z) = J(y,u) + \langle \lambda, c(y,u) \rangle_{\Lambda^*,\Lambda} + (z,u)_2.$$

Let $\bar{x} = (\bar{y}, \bar{u}) \in Y \times \mathcal{B}$ be a local solution of (2.1). Since $c_y(\bar{x})$ is surjective by (A2), the operator

$$h'(\bar{x}) = \begin{pmatrix} c_y(\bar{x}) & c_u(\bar{x}) \\ 0 & I \end{pmatrix} \in \mathcal{L}(X, \Lambda \times U)$$

is surjective and therefore *Robinson's constraint qualification* [15]

$$0 \in \text{int}(h(\bar{x}) + h_x(\bar{x}) \cdot X - \mathcal{C})$$

is satisfied. By standard optimality theory, see [5, Prop. 3.2], there exist $(\bar{\lambda}, \bar{z}) \in \Lambda^* \times U^*$ with

$$L_x(\bar{x}, \lambda, z) = (J_y + c_y^* \bar{\lambda}, J_u + c_u^* \bar{\lambda})(\bar{x}) + (0, \bar{z}) = 0, \quad h(\bar{x}) \in \mathcal{C}, \quad (\bar{\lambda}, \bar{z}) \in N_{\mathcal{C}}(h(\bar{x})) \tag{2.3}$$

with the normal cone

$$\begin{aligned} N_{\mathcal{C}}(h(\bar{x})) &:= \{(\lambda, z) \in \Lambda^* \times U^* \ : \ \langle \lambda, w - c(\bar{x}) \rangle_{\Lambda^*, \Lambda} + (z, v - \bar{u}) \le 0 \ \forall (w, v) \in \mathcal{C}\} \\ &= \{(\lambda, z) \in \Lambda^* \times U^* \ : \ (z, v - \bar{u}) \le 0 \ \forall v \in \mathcal{B}\} \\ &= \left\{(\lambda, z) \in \Lambda^* \times U^* \ : \ z|_{\{\bar{u}=a\}} \le 0, \quad z|_{\{\bar{u}=b\}} \ge 0, \ z|_{\{a<\bar{u}<b\}} = 0\right\}. \end{aligned}$$

Hence, using the splitting $\bar{z} = \bar{z}_b - \bar{z}_a$, $\bar{z}_b, \bar{z}_a \ge 0$, we can write (2.3) in the following form: there exist $\bar{\lambda} \in \Lambda^*$ and $\bar{z}_a, \bar{z}_b \in U^*$ such that with the Lagrangian

$$\ell(y, u, \lambda, z_a, z_b) = J(y, u) + \langle \lambda, c(y, u) \rangle_{\Lambda^*, \Lambda} - (z_a, u - a)_2 - (z_b, b - u)_2$$

the first order optimality conditions hold

$$\left\{ \begin{array}{l} \ell_y(\bar{x}, \bar{\lambda}, \bar{z}_a, \bar{z}_b) = J_y(\bar{x}) + c_y(\bar{x})^* \bar{\lambda} = 0, \\ \ell_u(\bar{x}, \bar{\lambda}, \bar{z}_a, \bar{z}_b) = J_u(\bar{x}) + c_u(\bar{x})^* \bar{\lambda} + \bar{z}_b - \bar{z}_a = 0, \\ c(\bar{x}) = 0, \quad a \le \bar{u} \le b, \\ (\bar{u} - a)\, \bar{z}_a = 0, \quad \bar{z}_a \ge 0, \\ (b - \bar{u})\, \bar{z}_b = 0, \quad \bar{z}_b \ge 0. \end{array} \right. \tag{2.4}$$

2.3. Barrier problem. Now consider the associated barrier-problem

$$\begin{aligned} &\min \ J_\mu(y, u) := J(y, u) - \mu \int_\Omega \ln(u - a)\, dx - \mu \int_\Omega \ln(b - u)\, dx \\ &\text{s.t.} \quad c(y, u) = 0, \quad a \le u \le b. \end{aligned} \tag{2.5}$$

PROPOSITION 2.3. *Under assumptions* (A1)–(A4) *problem (2.5) has for any $\mu > 0$ a solution.*

Proof. Take a minimizing sequence $(y(u_k), u_k)$. Since $(u_k) \subset \mathcal{B}$, it is bounded in L^∞ and has a weak*-convergent subsequence, for simplicity again denoted by (u_k), with limit $\bar{u} \in \mathcal{B}$. But by (A4) we have

$$J(y(u_k), u_k) \ge J(y(\bar{u}), \bar{u}). \tag{2.6}$$

Moreover, the barrier term satisfies

$$-\mu \int_\Omega \ln(u_k - a)\, dx - \mu \int_\Omega \ln(b - u_k)\, dx \ge -2\mu \text{leb}\,(\Omega) \ln(\|b - a\|_\infty).$$

This shows that

$$J_\mu(y(u_k), u_k) \ge J(y(\bar{u}), \bar{u}) - 2\mu \text{leb}\,(\Omega) \ln(\|b - a\|_\infty) \ge -M_1$$

with a constant $M_1 > 0$.

Moreover, we have $u_n \to \bar{u}$ in L^∞-weak* and thus also in L^2-weak. It remains to show that

$$u \in (\mathcal{B}, \|\cdot\|_2) \mapsto -\mu \int_\Omega \ln(u-a)\,dx - \mu \int_\Omega \ln(b-u)\,dx =: f_a(u) + f_b(u)$$

is lower semicontinuous w.r.t. weak convergence.

We consider only $f_a : u \mapsto -\mu \int_\Omega \ln(u-a)\,dx$, since $f_b : u \mapsto -\mu \int_\Omega \ln(b-u)\,dx$ can be treated analogously. Since $\mathcal{B} \subset L^2(\Omega)$ is closed and convex and $f_a : \mathcal{B} \mapsto \mathbb{R} \cup \{\infty\}$ is convex, it is sufficient to show that the mapping is lower semicontinuous w.r.t. strong convergence, see [11, Lem. 22.6] or [27, Prop. 38.7]. To this end, let $\mathcal{B} \ni v_k \to \bar{v}$ in L^2. We have to show

$$\liminf_{k\to\infty} f_a(v_k) \geq f_a(\bar{v}). \tag{2.7}$$

If $\liminf_{k\to\infty} f_a(v_k) = \infty$ this trivially holds.

Now consider the case $\lim_{k\to\infty} f_a(v_k) < \infty$. By selecting a subsequence $(v_k)_K$ we achieve

$$\lim_{K\ni k\to\infty} f_a(v_k) = \liminf_{k\to\infty} f_a(v_k) < \infty, \quad (v_k)_K \to \bar{v} \text{ almost everywhere on } \Omega.$$

In particular, there exists a constant $C > 0$ with $f_a(v_k) \leq C$ for all $k \in K$. We observe that

$$\begin{aligned} -\mu \ln(v_k - a) &= -\mu \ln(\max(v_k - a, 1)) - \mu \ln(\min(v_k - a, 1)) =: g_k + h_k, \\ -\mu \ln(\bar{v} - a) &= -\mu \ln(\max(\bar{v} - a, 1)) - \mu \ln(\min(\bar{v} - a, 1)) =: \bar{g} + \bar{h}. \end{aligned}$$

We have the estimate

$$|g_k - \bar{g}| \leq \mu |\max(v_k - a, 1) - \max(\bar{v} - a, 1)| \leq \mu |v_k - \bar{v}|$$

and thus

$$\lim_{K\ni k\to\infty} f_a(v_k) = \lim_{K\ni k\to\infty} \int_\Omega (g_k + h_k)\,dx = \int_\Omega \bar{g}\,dx + \lim_{k\to\infty} \int_\Omega h_k\,dx.$$

Moreover, $h_k \geq 0$, $|g_k| \leq \mu |v_k - a - 1|$ and, for $k \in K$,

$$0 \leq \int_\Omega h_k\,dx \leq C - \int_\Omega g_k\,dx \leq C + \mu \|v_k - a - 1\|_1 \leq C + C'.$$

From $(v_k)_K \to \bar{v}$ a.e. we obtain $(h_k)_K \to \bar{h}$ a.e.. Now the Lemma of Fatou (where we can work with $\lim$ instead of $\liminf$ here) yields $\bar{h} \in L^1(\Omega)$ and

$$0 \leq \int_\Omega \bar{h}\,dx = \int_\Omega \lim_{K\ni k\to\infty} h_k\,dx \leq \lim_{K\ni k\to\infty} \int_\Omega h_k\,dx \leq C + C'.$$

This concludes the proof that

$$\liminf_{k\to\infty} f_a(v_k) = \lim_{K\ni k\to\infty} f_a(v_k) = \liminf_{k\to\infty} \int_\Omega (g_k + h_k)\,dx \geq \int_\Omega (\bar{g} + \bar{h})\,dx = f_a(\bar{v}).$$

As mentioned before, the same holds for $v_k \to \bar{v}$ weakly, see Jost, Lemma 4.2.2.

Applying this to the minimizing sequence, we obtain together with (2.6)

$$\begin{aligned} \liminf_{k\to\infty} J_\mu(y(u_k), u_k) &= \liminf_{k\to\infty} J(y(u_k), u_k) + f_a(u_k) + f_b(u_k) \\ &\geq J(y(\bar{u}), \bar{u}) + f_a(\bar{u}) + f_b(\bar{u}) = J_\mu(y(\bar{u}), \bar{u}). \end{aligned}$$

Hence, $(y(\bar{u}), \bar{u})$ solves (2.5). □

REMARK 2.4. It is important to point out that the proof would be easier if we could assume that the solution lies in the L^∞-interior of $\mathcal{B}$. Since, however, the case that $\bar{u}$ touches the bounds on a null set cannot be excluded, we need alternative proof techniques as given above. If the problem has special structure, e.g., as in [14], it is sometimes possible to prove that $\bar{u}$ lies in the L^∞-interior of $\mathcal{B}$ and then a less technical argumentation can be used, see, e.g., section 3 of [14]. It is obvious that $\{\bar{u} = a\}$ and $\{\bar{u} = b\}$ have measure zero, since $J_\mu(\bar{y}, \bar{u}) < \infty$. Therefore, it is easy to derive the following necessary optimality conditions for (2.5).

LEMMA 2.5. *Let assumptions* (A1)–(A4) *hold and let* $x_\mu = (y_\mu, u_\mu)$ *be a local solution of (2.5),* $\mu > 0$*. Then there is* $\lambda_\mu \in \Lambda^*$ *such that*

$$\left\{\begin{array}{l} J_y(x_\mu) + c_y(x_\mu)^*\lambda_\mu = 0, \\ J_u(x_\mu) + c_u(x_\mu)^*\lambda_\mu + \dfrac{\mu}{u_\mu} - \dfrac{\mu}{u_\mu - a} = 0, \\ c(x_\mu) = 0, \quad a < u_\mu < b. \end{array}\right. \tag{2.8}$$

REMARK 2.6. By introducing the artificial variables $z_{a,\mu} = \frac{\mu}{u_\mu - a}$ and $z_{b,\mu} = \frac{\mu}{b - u_\mu}$ we can write (2.8) as the perturbed KKT-conditions

$$\left\{\begin{array}{l} \ell_y(x_\mu, \lambda_\mu, z_{a,\mu}, z_{b,\mu}) = J_y(x_\mu) + c_y(x_\mu)^*\lambda_\mu = 0, \\ \ell_u(x_\mu, \lambda_\mu, z_{a,\mu}, z_{b,\mu}) = J_u(x_\mu) + c_u(x_\mu)^*\lambda_\mu + z_{b,\mu} - z_{a,\mu} = 0, \\ c(x_\mu) = 0, \quad a \le u_\mu \le b, \\ (u_\mu - a)\, z_{a,\mu} = \mu, \quad z_{a,\mu} \ge 0, \\ (b - u_\mu)\, z_{b,\mu} = \mu, \quad z_{b,\mu} \ge 0. \end{array}\right. \tag{2.9}$$

We call the solution set (2.9) parameterized by $\mu > 0$ *central path*. We will see that under appropriate assumptions the central path is actually a Hölder-continuous curve that converges for $\mu \to 0$ to a solution of (2.1).□

Proof. By (A3) there exists for any $u \in \mathcal{D}$ a unique solution $y = y(u) \in Y$ of $c(y, u) = 0$. By (A2) and the implicit function theorem the mapping $u \in (\mathcal{D}, \|\cdot\|_U) \mapsto y(u) \in Y$ is continuously differentiable with

$$c_y y_u = -c_u.$$

Thus the reduced objective functional $u \in (\mathcal{D}, \|\cdot\|_U) \mapsto \hat{J}(u)$ is continuously differentiable with derivative

$$\begin{aligned} \hat{J}_u(u) &= -\langle J_y(x), c_y(x)^{-1} c_u(x) \,\cdot\, \rangle_{Y^*,Y} + J_u(x) \\ &= -c_u(x)^* c_y(x)^{-*} J_y(x) + J_u(x) \\ &= -c_u(x)^* c_y(x)^{-*} J_y(x) + J_u(x), \end{aligned}$$

where $x = (y(u), u)$. Let $(y(u_\mu), u_\mu)$ be the solution of (2.5). With the unique solution $\lambda_\mu \in \Lambda^*$ of

$$J_y(x_\mu) + c_y(x_\mu)^*\lambda_\mu = 0$$

we have

$$\hat{J}_u(u - \mu) = c_u(x_\mu)^*\lambda_\mu + J_u(x_\mu). \tag{2.10}$$

We show that

$$w := \hat{J}_u(u_\mu) - \frac{\mu}{u_\mu - a} + \frac{\mu}{b - u_\mu} = 0 \quad \text{a.e.}$$

We know that $a < u_\mu < b$ almost everywhere. The sets

$$M_k := \{a + 1/k \le u_\mu \le b - 1/k\}$$

are monotone increasing with $\bigcup_{k=1}^\infty M_k = \Omega \setminus N$ with a set N of measure zero. Let $v \in L^\infty(\Omega)$ be arbitrary, then $v_k := v1_{M_k} \to v$ in $U = L^p(\Omega)$, since $p < \infty$. For all $t \in (-\rho, \rho)$, $\rho > 0$ small enough, we have $a + 1/(2k) \le u_\mu + tv_k \le b - 1/(2k)$ and therefore the function

$$\begin{aligned} h_k : t \in (-\rho, \rho) \mapsto & J_\mu(y(u_\mu + tv_k), u_\mu + tv_k) \\ &= \hat{J}(u_\mu + tv_k) - \mu \int_\Omega \ln(b - (u_\mu + tv_k))\, dx - \mu \int_\Omega \ln((u_\mu + tv_k) - a) \end{aligned}$$

is continuously differentiable with

$$h_k'(t) = \left\langle \hat{J}_u(u_\mu + tv_k) + \frac{\mu}{b - (u_\mu + tv_k)} - \frac{\mu}{(u_\mu + tv_k) - a}, v_k \right\rangle_2 .$$

Since $(y(u_\mu), u_\mu)$ is optimal for (2.5) and $u_\mu + tv_k \in \mathcal{B}$ for $t \in (-\rho, \rho)$, the function h_k has a minimum at $t = 0$ and thus

$$0 = h_k'(0) = \left\langle \hat{J}_u(u_\mu) + \frac{\mu}{b - u_\mu} - \frac{\mu}{u_\mu - a}, v_k \right\rangle_2 .$$

Taking the limit $k \to \infty$ we obtain

$$\left\langle \hat{J}_u(u_\mu) + \frac{\mu}{b - u_\mu} - \frac{\mu}{u_\mu - a}, v \right\rangle_2 = 0.$$

This holds for all $v \in L^\infty(\Omega)$ and by density for all $v \in U$. We deduce with (2.10) that

$$c_u(x_\mu)^* \lambda_\mu + J_u(x_\mu) + \frac{\mu}{b - u_\mu} - \frac{\mu}{u_\mu - a} = 0.$$

□

REMARK 2.7. For the special case of linear elliptic control problems, the previous results were shown in a different way in [14]. The control problem in [14] satisfies our assumption $(\mathrm{A5})_q$ below with $q = \infty$. In this particular case the solution of the barrier problem (2.5) lies in the interior of $\mathcal{B}$, see Corollary 4.4 below, and $\bar{z}_a, \bar{z}_b$ are bounded in L^∞. The analysis in [14] makes essential use of this fact.

In this paper we cover the more general setting that $\bar{z}_a, \bar{z}_b$ are only bounded in L^q for some $q > p$. This is of essential interest to cover state equations where the state or adjoint equation does not allow for a priori estimates in L^∞. In the latter case solutions of (2.5) can touch the boundary of $\mathcal{B}$ on a zero set and are thus no interior points in the classical sense. Nevertheless, we will see that also in this setting interior-point methods with a projection are convergent, since – roughly speaking – the measure of the set where the solution of (2.5) has distance $\le \varepsilon$ to the boundary of $\mathcal{B}$ tends to zero as $\varepsilon \searrow 0$. The analysis is considerably more involved than for the case $(\mathrm{A5})_\infty$. □

3. A function space setting. Unfortunately, it is not possible to work with soft analysis only. Rather, we need a carefully adjusted function space setting, where a typical requirement will be that a continuous (or differentiable) mapping $h : X_1 \to Y_1$ also defines a mapping $h : X_2 \to Y_2$ ¿¿from a stronger space $X_2 \subset X_1$ to a stronger space $Y_2 \subset Y_1$. For instance, as a trivial example, the identity mapping $X_1 \ni x \mapsto x \in X_1$ induces the identity mapping $X_2 \ni x \mapsto x \in X_2$ for any stronger space $X_2 \subset X_1$.

We make the following assumptions, which are satisfied for many elliptic and parabolic optimal control problems, see [17, 18] and the references therein.

(A5)$_q$ There are $q \in (p, \infty]$ and Banach spaces $\Sigma \subset \Lambda^*$, $V \subset Y^*$ such that the following holds:

1. The mapping

$$(y, u, \lambda) \in Y \times L^q(\Omega) \times \Sigma \mapsto \ell_u(y, u, \lambda, 0, 0) \in L^q(\Omega)$$

   is differentiable and its derivative is Lipschitz continuous on bounded subsets of $Y \times \mathcal{B} \times \Sigma$.
2. The mapping

$$(y, u) \in Y \times U \mapsto J_y(y, u) \in V$$

   is differentiable and its derivative is Lipschitz continuous on bounded subsets of $Y \times \mathcal{B}$.
3. The operator

$$(y, u) \in Y \times U \mapsto c_y^*(y, u) \in \mathcal{L}(\Sigma, V)$$

   is differentiable and its derivative is Lipschitz continuous on bounded subsets of $Y \times \mathcal{B}$.
4. The following mappings are continuous and uniformly bounded on bounded sets:

$$\begin{aligned}
&(y, u) \in Y \times (\mathcal{B}, \|\cdot\|_q) \mapsto J_u(y, u) \in L^q(\Omega), \\
&(y, u) \in Y \times U \mapsto c_u^*(y, u) \in \mathcal{L}(\Sigma, L^q(\Omega)), \\
&(y, u) \in Y \times U \mapsto c_y^{-*}(y, u) \in \mathcal{L}(V, \Sigma), \\
&(y, u, \lambda) \in Y \times U \times \Sigma \mapsto \ell_{uy}(y, u, \lambda, 0, 0) \in \mathcal{L}(Y, L^q(\Omega)), \\
&(y, u, \lambda) \in Y \times U \times \Sigma \mapsto \ell_{yu}(y, u) \in \mathcal{L}(L^2(\Omega), V), \\
&(y, u, \lambda) \in Y \times U \times \Sigma \mapsto \ell_{uu}(y, u) \in \mathcal{L}(L^t(\Omega), L^t(\Omega)), \quad t \in [2, q].
\end{aligned}$$

5. The reduced gradient has the structure

$$\ell_u(y, u, \lambda, 0, 0) = \beta(u) + \hat{g}_s(y, u, \lambda), \quad \beta \in C^1(\mathbb{R}),\ \beta' \geq \alpha_0 > 0,$$

   where

$$(y, u, \lambda) \in Y \times U \times \Sigma \mapsto \hat{g}_s(y, u, \lambda) \in L^q(\Omega)$$

   is Lipschitz continuous on bounded sets.
6. The reduced Hessian

$$\hat{H}(y, u, \lambda) := \ell_{uu} + c_u^* c_y^{-*} \ell_{yy} c_y^{-1} c_u - c_u^* c_y^{-*} \ell_{yu} - \ell_{uy} c_y^{-1} c_u \tag{3.1}$$

   has the structure

$$\hat{H}(y, u, \lambda) = \beta'(u) I + \hat{H}_s(y, u, \lambda),$$

   where $(y, u, \lambda) \in Y \times U \times \Sigma \mapsto \hat{H}_s(y, u, \lambda) \in \mathcal{L}(L^2, L^q)$ is uniformly bounded on bounded subsets.

REMARK 3.1. *We point out that usually, (A5)$_{q_2}$ implies (A5)$_{q_1}$ for $q_2 > q_1$. In fact, since $L^{q_2} \subset L^{q_1}$, only part 1 and the first condition in part 4 can be critical, but for the examples to follow, they are not.*

3.1. Verification for the elliptic control problem.

3.1.1. First setting. In the first setting we can verify (A5$_q$) for any $q > 2$ such that $Y = H_0^1(\Omega)$ is continuously embedded in $L^q(\Omega)$. We do not need the additional spaces V and Σ, since we just can choose $V = Y^* = H^{-1}(\Omega)$ and $\Sigma = \Lambda^* = H_0^1(\Omega) = Y$.

We have

$$\begin{aligned}
\ell(y, u, \lambda, z_a, z_b) &= \frac{1}{2}\|y - y_d\|_2^2 + \frac{\alpha}{2}\|u\|_2^2 \\
&\quad + \langle \lambda, Ay - u\rangle - (z_a, u - a)_2 + (z_b, u - b)_2, \\
J_y(y, u) &= y - y_d, \quad J_u(y, u) = \alpha u, \\
\ell_y(y, u, \lambda, z_a, z_b) &= J_y(y, u) + A^*\lambda = y - y_d + A\lambda, \\
\ell_u(y, u, \lambda, z_a, z_b) &= \alpha u - \lambda - z_a + z_b, \\
\ell_{yu}(y, u, \lambda, z_a, z_b) &= 0, \\
\ell_{uy}(y, u, \lambda, z_a, z_b) &= 0, \\
\ell_{yy}(y, u, \lambda, z_a, z_b) &= I_{Y,Y^*}, \\
\ell_{uu}(y, u, \lambda, z_a, z_b) &= \alpha I.
\end{aligned}$$

Therefore, (A5$_q$) is a direct consequence of the following observations:

1. $\ell_u(y, u, \lambda, 0, 0) = \alpha u - \lambda \in L^q(\Omega) + \Sigma = L^q(\Omega)$.
2. $J_y(y, u) = y - y_d \in L^2 \subset H^{-1} = V$.
3. $c_y^*(y, u) = A \in \mathcal{L}(H_0^1, H^{-1}) = \mathcal{L}(\Sigma, V)$.
4. $J_u(y, u) = \alpha u$,
   $c_u^*(y, u) = I_{\Lambda^*,U^*} = I_{H_0^1,L^2} \in \mathcal{L}(\Sigma, L^q(\Omega))$,
   $c_y^{-*}(y, u) = A^{-1} \in \mathcal{L}(H^{-1}, H_0^1) = \mathcal{L}(V, \Sigma)$,
   $\ell_{uy}(y, u, \lambda, 0, 0) = 0$,
   $\ell_{yu}(y, u) = 0$,
   $\ell_{uu}(y, u) = \alpha I$.
5. $\ell_u(y, u, \lambda, 0, 0) = \alpha u - \lambda = \beta(u) + \hat{g}_s(y, u, \lambda)$
   with $\beta(t) = \alpha t$, $\beta'(t) = \alpha$, $\hat{g}_s(y, u, \lambda) = -\lambda$.
   Hence, $\beta \in C^1(\mathbb{R})$, $\beta'(t) = \alpha \geq \alpha_0$ for any $\alpha_0 \in (0, \alpha]$, and $\hat{g}_s(y, u, \lambda) = -\lambda \in H_0^1 \subset L^q(\Omega)$.
6. $\hat{H}(y, u, \lambda) = \alpha I + I_{H_0^1,L^2} A^{-1} I_{H_0^1,H^{-1}} A^{-1} I_{L^2,H^{-1}} = \alpha I + A^{-2} = \beta'(u)I + \hat{H}_s(y, u, \lambda)$,
   where $\hat{H}_s(y, u, \lambda) = A^{-2} \in \mathcal{L}(H^{-1}, H_0^1) \subset \mathcal{L}(L^2, L^q)$ is constant and thus uniformly bounded on bounded subsets.

3.1.2. Second setting. Here, we choose $V = L^2$, $\Sigma = \tilde{Y}$, and verify (A5$_q$) for any $q \in (2, \infty]$ satisfying $\tilde{Y} = H_0^1 \cap H^2 \subset L^q$ (continuous embedding). In particular, for $n \leq 3$, we have $\tilde{Y} \subset L^\infty$.

It is quite obvious that the operator $A \in \mathcal{L}(H_0^1, H^{-1})$ from the first setting is the unique extension of $\tilde{A} \in \mathcal{L}(\tilde{Y}, L^2) \subset \mathcal{L}(\tilde{Y}, H^{-1})$ to $H_0^1 \supset \tilde{Y}$. Therefore, $\tilde{A}^* \in \mathcal{L}(L^2, \tilde{Y}^*)$ is the unique extension of $A^* = A \in \mathcal{L}(H_0^1, H^{-1}) \subset \mathcal{L}(H_0^1, \tilde{Y}^*)$ to $L^2 \supset H_0^1$. Since, as said, $A = A^*$ is the unique extension of $\tilde{A}$, this shows that $\tilde{A}^* \in \mathcal{L}(L^2, \tilde{Y}^*)$ is the unique extension of $\tilde{A} \in \mathcal{L}(\tilde{Y}, L^2) \subset \mathcal{L}(\tilde{Y}, \tilde{Y}^*)$ to L^2. Hence,

$$y \in \tilde{Y} \mapsto \tilde{A}^* y \in L^2 \text{ is continuous linear and boundedly invertible in } \mathcal{L}(\tilde{Y}, L^2). \tag{3.2}$$

The derivatives can be computed similar to the first setting. The validity of (A5$_q$) follows from

1. $\ell_u(y, u, \lambda, 0, 0) = \alpha u - \lambda \in L^q(\Omega) + \Sigma = L^q(\Omega) + \tilde{Y} = L^q(\Omega)$.
2. $J_y(y, u) = y - y_d \in L^2 = V$.

3. $c_y^*(y,u) = \tilde{A}^* \in \mathcal{L}(\tilde{Y}, L^2) = \mathcal{L}(\Sigma, V)$ (see (3.2)).

4. $J_u(y,u) = \alpha u$,
$c_u^*(y,u) = I_{\Lambda^*,U^*} = I_{L^2} \in \mathcal{L}(\Sigma, L^q(\Omega))$,
$c_y^{-*}(y,u) = \tilde{A}^{-*} \in \mathcal{L}(L^2, \tilde{Y}) = \mathcal{L}(V, \Sigma)$ (see (3.2)),
$\ell_{uy}(y,u,\lambda,0,0) = 0$,
$\ell_{yu}(y,u) = 0$,
$\ell_{uu}(y,u) = \alpha I$.

5. $\ell_u(y,u,\lambda,0,0) = \alpha u - \lambda = \beta(u) + \hat{g}_s(y,u,\lambda)$
with $\beta(u) = \alpha t$, $\beta'(t) = \alpha$, $\hat{g}_s(y,u,\lambda) = -\lambda$.
Hence, $\beta \in C^1(\mathbb{R})$, $\beta'(t) = \alpha \geq \alpha_0$ for any $\alpha_0 \in (0, \alpha]$, and

$$\hat{g}_s(y,u,\lambda) = -\lambda \in \Sigma = \tilde{Y} \subset L^q(\Omega) \quad \text{for } \lambda \in \Sigma.$$

6. By (3.2), $\hat{H}(y,u,\lambda) = \alpha I + \tilde{A}^{-*} I_{\tilde{Y},\tilde{Y}^*} \tilde{A}^{-1} = \alpha I + \tilde{A}^{-2} = \beta'(u) I + \hat{H}_s(y,u,\lambda)$,
where $\hat{H}_s(y,u,\lambda) = \tilde{A}^{-2} \in \mathcal{L}(L^2, \tilde{Y}) \subset \mathcal{L}(L^2, L^q)$ is constant and thus uniformly bounded on bounded subsets.

4. Properties of the central path. We study next the regularity of the dual variables $z_{a,\mu}, z_{b,\mu}$ on the central path.

LEMMA 4.1. *Let* (A1)–(A5)$_q$ *hold and let* $(y_\mu, u_\mu, \lambda_\mu, z_{a,\mu}, z_{b,\mu})$ *be a solution of (2.9). Then there holds* $\lambda_\mu \in \Sigma$*,*

$$0 \leq z_{a,\mu}, z_{b,\mu} \leq \max(3\mu/\nu, 2|J_u(x_\mu) + c_u(x_\mu)^*\lambda_\mu|), \tag{4.1}$$

and thus with (A5)$_q$

$$\|z_{a,\mu}\|_q, \|z_{b,\mu}\|_q \leq \|\max(3\mu/\nu, 2|J_u(x_\mu) + c_u(x_\mu)^*\lambda_\mu|)\|_q < \infty. \tag{4.2}$$

Proof. From the first equation in (2.9) we see that

$$\lambda_\mu = -c_y(x_\mu)^{-*} J_y(x_\mu) \in c_y(x_\mu)^{-*} V \subset \Sigma.$$

We have

$$z_{a,\mu} = \frac{\mu}{u_\mu - a}, \quad z_{b,\mu} = \frac{\mu}{b - u_\mu}.$$

This yields on the set $M = \{u_\mu - a \leq (b - u_\mu)/2\}$ the estimate $z_{a,\mu}|_M \geq 2 z_{b,\mu}|_M$ and thus by (2.9)

$$0 \leq \frac{1}{2} z_{a,\mu}|_M \leq z_{a,\mu}|_M - z_{b,\mu}|_M = (J_u(x_\mu) + c_u(x_\mu)^*\lambda_\mu)|_M.$$

On the complement $M^c := \Omega \setminus M$ we have

$$\frac{3}{2}(u_\mu - a)|_{M^c} \geq \frac{1}{2}(b-a)|_{M^c} \geq \frac{1}{2}\nu$$

and thus $(u_\mu - a)|_{M^c} \geq \nu/3$. Both cases together prove (4.1) for $z_{a,\mu}$. The estimate for $z_{b,\mu}$ is obtained in the same way. Since $\lambda_\mu \in \Sigma$, the right hand side of (4.1) is in L^q by (A5)$_q$ and thus (4.2) is obvious. ∎

We introduce for $s \in [1,\infty]$ the function spaces

$$W_s := Y \times L^s \times \Sigma \times L^s \times L^s,$$
$$W_s' := V \times L^s \times \Lambda \times L^s \times L^s,$$

equipped with the norms

$$\|(y,u,\lambda,z_a,z_b)\|_{W_s} = \|y\|_Y + \|\lambda\|_\Sigma + \left\|\sqrt{u^2+z_a^2+z_b^2}\right\|_s,$$
$$\|(y,u,v,z_a,z_b)\|_{W_s'} = \|y\|_V + \|v\|_\Lambda + \left\|\sqrt{u^2+z_a^2+z_b^2}\right\|_s.$$

REMARK 4.2. The choice of the Euclidean norm for $(u(\xi), z_a(\xi), z_b(\xi)) \in \mathbb{R}^3$ will be convenient, since we will later use a pointwise orthogonal projection of these components with respect to the Euclidean inner product on $\mathbb{R}^3$.□

As a direct consequence of the previous lemma all solutions of the perturbed optimality conditions (2.9) are contained in a bounded set of W_q.

COROLLARY 4.3. *Let* (A1)–(A5)$_q$ *hold. Then for any* $\mu_0 > 0$ *there exists a constant* $C_{\mu_0} > 0$ *such that for all* $0 < \mu \le \mu_0$ *any solution* $w_\mu = (y_\mu, u_\mu, \lambda_\mu, z_{a,\mu}, z_{b,\mu})$ *of (2.9) is in* W_q *and*

$$\|w_\mu\|_{W_q} \le C_{\mu_0}.$$

Proof. Since $u_\mu \in \mathcal{B}$ we have $\|u_\mu\|_q \le \|a\|_q + \|b\|_q =: C_u$. By Remark 2.1 $u \in U \to y(u) \in Y$ is Lipschitz continuous on $\mathcal{B}$ and therefore $\|y_\mu\|_Y = \|y(u_\mu)\|_Y \le C_y$ with a constant C_y. Now (2.9) yields

$$\lambda_\mu = -c_y(x_\mu)^{-*} J_y(x_\mu)$$

and by (A5)$_q$ the right hand side is uniformly bounded in Σ, since x_μ lies in a bounded subset of $Y \times \mathcal{B}$. Finally, this implies with (A5)$_q$ that the right hand side of (4.2) is uniformly bounded. The proof is complete. □

If (A5)$_\infty$ is satisfied then we can deduce immediately that solutions of the barrier problem are true interior points. In fact, we have the simple

COROLLARY 4.4. *Let* (A1)–(A5)$_\infty$ *hold. Then for any* $\mu_0 > 0$ *there exists a constant* $C_{\mu_0} > 0$ *such that for all* $0 < \mu \le \mu_0$ *any solution* $w_\mu = (y_\mu, u_\mu, \lambda_\mu, z_{a,\mu}, z_{b,\mu})$ *of (2.9) satisfies*

$$u_\mu - a \ge \frac{\mu}{C_{\mu_0}}, \quad b - u_\mu \ge \frac{\mu}{C_{\mu_0}}.$$

Proof. Corollary 4.3 yields a constant $C_{\mu_0} > 0$ with

$$\|z_{a,\mu}\|_\infty, \|z_{b,\mu}\|_\infty \le \|w_\mu\|_{W_\infty} \le C_{\mu_0}.$$

Now the last two equations in (2.9) yield

$$u_\mu - a = \frac{\mu}{z_{a,\mu}} \ge \frac{\mu}{C_{\mu_0}}, \quad b - u_\mu = \frac{\mu}{z_{b,\mu}} \ge \frac{\mu}{C_{\mu_0}}.$$

□

REMARK 4.5. For linear elliptic control problems, which satisfy (A1)–(A5)$_\infty$, this result was shown in [14], where it is used to prove the existence of solutions for (2.5). We used a different proof to cover also the more general case that (A5)$_q$ holds only for some $q < \infty$. □

We show next, that the dual variables $\lambda_\mu, z_{a,\mu}, z_{b,\mu}$ depend continuously on the primal variables y_μ, u_μ.

LEMMA 4.6. *Let* (A1)–(A5)$_q$ *hold and let* $(y_\mu, u_\mu, \lambda_\mu, z_{a,\mu}, z_{b,\mu})$ *be a solution of (2.9). Then*

$$\lambda_\mu = -c_y(x_\mu)^{-*} J_y(x_\mu)$$

and for any measurable sets $M, N \subset \Omega$ *one has*

$$\begin{aligned} z_{a,\mu}|_M &= \left.\frac{\mu}{u_\mu - a}\right|_M, \\ z_{b,\mu}|_N &= \left.\frac{\mu}{b - u_\mu}\right|_N, \\ z_{a,\mu}|_N &= (z_{b,\mu} + J_u(x_\mu) + c_u(x_\mu)^* \lambda_\mu)|_N \\ z_{b,\mu}|_M &= (z_{a,\mu} - J_u(x_\mu) - c_u(x_\mu)^* \lambda_\mu)|_M. \end{aligned} \tag{4.3}$$

Moreover, if $u_\eta \to u_\mu$ *in* $L^q(\Omega)$ *as* $\eta \to \mu$ *then*

$$(u_\eta, y_\eta, \lambda_\eta, z_{a,\eta}, z_{b,\eta}) \to (u_\mu, y_\mu, \lambda_\mu, z_{a,\mu}, z_{b,\mu}) \quad \textit{in } W_q.$$

Proof. The equations for λ_μ, $z_{a,\mu}$, and $z_{b,\mu}$ follow directly from (2.9).

Now assume that $u_\eta \to u_\mu$ in $L^q(\Omega)$ as $\eta \to \mu$. Then $y_\eta = y(u_\eta) \to y_\mu = y(u_\mu)$ in Y follows from (A1)–(A3), see Remark 2.1. Moreover,

$$\lambda_\eta = -c_y(x_\eta)^{-*} J_y(x_\eta) \to \lambda_\mu = -c_y(x_\mu)^{-*} J_y(x_\mu) \quad \text{in } \Sigma$$

is a direct consequence of (A5)$_q$.

Finally, we know by Lemma 4.1 that $z_{a,\eta}, z_{b,\eta} \in L^q(\Omega)$ for all $\eta > 0$.

Now we use the formulas (4.3) with

$$\begin{aligned} M = M_\eta &:= \{u_\eta \ge a + (b-a)/4,\ u_\mu \ge a + (b-a)/4\}, \\ N = N_\eta &:= \{u_\eta \le a + 3(b-a)/4,\ u_\mu \le a + 3(b-a)/4\} \setminus M_\eta. \end{aligned}$$

Then we obtain on M_η by (4.3)

$$\begin{aligned} |(z_{a,\eta} - z_{a,\mu})_{M_\eta}| = \left| \left(\frac{\eta}{u_\eta - a} - \frac{\mu}{u_\mu - a} \right)_{M_\eta} \right| &\le \left. \frac{|\eta - \mu|}{u_\eta - a} \right|_{M_\eta} + \left. \frac{\mu |u_\eta - u_\mu|}{(u_\eta - a)(u_\mu - a)} \right|_{M_\eta} \\ &\le \frac{4|\eta - \mu|}{\nu} + \frac{16\mu}{\nu^2} |(u_\eta - u_\mu)_{M_\eta}| \end{aligned}$$

and thus

$$\|(z_{a,\eta} - z_{a,\mu})|_{M_\eta}\|_q \le C(|\eta - \mu| + \|(u_\eta - u_\mu)|_{M_\eta}\|_q) \to 0 \quad \text{as } \eta \to \mu.$$

Now (4.3) yields with (A5)$_q$

$$\begin{aligned} \|(z_{b,\eta} - z_{b,\mu})|_{M_\eta}\|_q \le{}& \|(z_{a,\eta} - z_{a,\mu})|_{M_\eta}\|_q + \|J_u(x_\eta) - J_u(x_\mu)\|_q \\ &+ \|c_u(x_\eta)^* \lambda_\eta - c_u(x_\mu)^* \lambda_\mu)\|_q \to 0 \quad \text{as } \eta \to \mu. \end{aligned}$$

In the same way we obtain

$$\|(z_{a,\eta} - z_{a,\mu})|_{N_\eta}\|_q + \|(z_{b,\eta} - z_{b,\mu})|_{N_\eta}\|_q \to 0 \quad \text{as } \eta \to \mu.$$

Next, we show that for $\mu/3 < \eta < 3\mu$, there holds on $J_\eta := \Omega \setminus (M_\eta \cup N_\eta)$:

(4.4) $$\operatorname{sgn}(z_{a,\eta} - z_{a,\mu})|_{J_\eta} = -\operatorname{sgn}(z_{b,\eta} - z_{b,\mu})|_{J_\eta}.$$

In fact, consider

$$J_\eta^1 = \{u_\eta > a + 3(b-a)/4,\ u_\mu < a + (b-a)/4\}.$$

Then

$$(z_{a,\eta} - z_{a,\mu})_{J_\eta^1} = \left(\frac{\eta}{u_\eta - a} - \frac{\mu}{u_\mu - a}\right)_{J_\eta^1} \leq \left(\frac{4\eta}{3} - 4\mu\right)\frac{1}{(b-a)_{J_\eta^1}} < 0$$

for $\eta < 3\mu$.

$$(z_{b,\eta} - z_{b,\mu})_{J_\eta^1} = \left(\frac{\eta}{b - u_\eta} - \frac{\mu}{b - u_\mu}\right)_{J_\eta^1} \geq \left(4\eta - \frac{4\mu}{3}\right)\frac{1}{(b-a)_{J_\eta^1}} > 0$$

for $\eta > \mu/3$.

Now, consider

$$J_\eta^2 = \{u_\eta < a + (b-a)/4,\ u_\mu > a + 3(b-a)/4\}.$$

Then

$$(z_{a,\eta} - z_{a,\mu})_{J_\eta^1} = \left(\frac{\eta}{u_\eta - a} - \frac{\mu}{u_\mu - a}\right)_{J_\eta^1} \geq \left(4\eta - \frac{4\mu}{3}\right)\frac{1}{(b-a)_{J_\eta^2}} > 0$$

for $\eta > \mu/3$.

$$(z_{b,\eta} - z_{b,\mu})_{J_\eta^1} = \left(\frac{\eta}{b - u_\eta} - \frac{\mu}{b - u_\mu}\right)_{J_\eta^1} \leq \left(\frac{4\eta}{3} - 4\mu\right)\frac{1}{(b-a)_{J_\eta^2}} < 0$$

for $\eta < 3\mu$.

Thus, using (4.4), the difference of the second equation in (2.9) for η and μ, respectively, yields

$$\begin{aligned}&\left\|(z_{b,\eta} - z_{b,\mu})|_{J_\eta}\right\|_q + \left\|(z_{a,\eta} - z_{a,\mu})|_{J_\eta}\right\|_q = \left\|(z_{b,\eta} - z_{b,\mu} + z_{a,\mu} - z_{a,\eta})|_{J_\eta}\right\|_q \\ &\leq \left\|J_u(x_\eta) - J_u(x_\mu)\right\|_q + \left\|c_u(x_\eta)^*\lambda_\eta - c_u(x_\mu)^*\lambda_\mu\right\|_q \to 0 \quad \text{as } \eta \to \mu.\end{aligned}$$

Since $M_\eta \cup N_\eta \cup J_\eta = \Omega$, we have shown that

$$\left\|z_{a,\eta} - z_{a,\mu}\right\|_q + \left\|z_{b,\eta} - z_{b,\mu}\right\|_q \to 0 \quad \text{as } \eta \to \mu$$

which concludes the proof. □

5. Analysis of the primal-dual Newton system. Throughout this section we assume that (A1)–(A5)$_q$ with some $q > p$ hold.

5.1. Primal-dual Newton system. For convenience, we will use the abbreviations

$$u_a = u - a, \quad u_b = b - u.$$

The formal application of Newton's method to the perturbed KKT-system (2.9) yields with the multiplication operators

$$Z_a := z_a \cdot I, \quad Z_b := z_b \cdot I, \quad U_a := (u-a) \cdot I, \quad U_b := (b-u) \cdot I$$

the primal-dual Newton system

$$\begin{pmatrix} \ell_{yy} & \ell_{yu} & c_y^* & 0 & 0 \\ \ell_{uy} & \ell_{uu} & c_u^* & -I & I \\ c_y & c_u & 0 & 0 & 0 \\ 0 & Z_a & 0 & U_a & 0 \\ 0 & -Z_b & 0 & 0 & U_b \end{pmatrix} \begin{pmatrix} s_y \\ s_u \\ s_\lambda \\ s_a \\ s_b \end{pmatrix} = - \begin{pmatrix} \ell_y(y,u,\lambda,z_a,z_b) \\ \ell_u(y,u,\lambda,z_a,z_b) \\ c(y,u) \\ z_a(u-a)-\mu \\ z_b(b-u)-\mu \end{pmatrix}.$$

We write this briefly as

$$DF_\mu(w)\, s = -F_\mu(w).$$

To ensure a certain quality of the primal-dual Newton step, we will keep the iteration in the following wide neighborhood of the central path

$$\begin{aligned} N_{-\infty,q}(\mu) := \Bigg\{ & (y,u,\lambda,z_a,z_b) \in Y \times \mathcal{B} \times \Sigma \times L^q \times L^q \ : \\ & (u-a)z_a \geq \gamma\mu, \ \ (b-u)z_b \geq \gamma\mu, \\ & z_a|_{\{u>(b+a)/2\}} \leq \frac{2\max(\mu_{-\infty},\mu/\gamma)}{\nu}, \ \ z_b|_{\{u<(b+a)/2\}} \leq \frac{2\max(\mu_{-\infty},\mu/\gamma)}{\nu}, \\ & \min(z_a|_{\{u=(b+a)/2\}}, z_b|_{\{u=(b+a)/2\}}) \leq \frac{2\max(\mu_{-\infty},\mu/\gamma)}{\nu} \Bigg\} \end{aligned}$$

with constants $\gamma \in (0,1)$, $\mu_{-\infty} > 0$. By Corollary 4.3 any solution of (2.9) is contained in $N_{-\infty,q}(\mu)$.

We note that $N_{-\infty,q}(\mu)$ is similar to the usual wide neighborhood used by many finite dimensional primal-dual interior-point methods. The difference are the upper bounds on $z_a|_{\{u\geq(b+a)/2\}}$ and $z_b|_{\{u\leq(b+a)/2\}}$, which excludes too large multipliers at points with strongly inactive bound constraints. Our neighborhood is still a wide neighborhood, since it approaches for $\mu \searrow 0$ the usual wide neighborhood and since for $\gamma \searrow 0$ it exhausts the set $\{(y,u,\lambda,z_a,z_b) \in Y \times \mathcal{B} \times \Sigma \times L^q \times L^q \ : \ z_a, z_b \geq 0\}$.

The neighborhood poses no constraints on y and λ, but it poses pointwise a.e. constraints of the form $(u(\xi), z_a(\xi), z_b(\xi)) \in N_\xi \subset \mathbb{R}^3$ on the triple (u, z_a, z_b). A sketch of the set N_ξ is depicted in Figure 5.1. It is the union of two congruent convex sets. The boundary is composed of planes and of hyperbolas that are extruded along the z_a- and z_b-axis, respectively, see also Remark 7.2.

5.2. Regularity of the primal-dual Newton system. We will now show that under a coercivity condition for the reduced Hessian $\hat{H}$ and under assumptions (A1)-(A5)$_q$ for all $w \in N_{-\infty,q}(\mu)$ the operator $DF_\mu(w)$ has a right inverse $DF_\mu(w)^{-1} \in \mathcal{L}(W_t', W_t)$, $t \in [p,q]$, with

$$\|DF_\mu(w)^{-1}\|_{W_t',W_t} \leq \frac{C}{\min(1,\sqrt{\mu})}.$$

Moreover, if we premultiply $DF_\mu(w)$ by the scaling operator

$$S(w) = \begin{pmatrix} I & & & & \\ & I & & & \\ & & I & & \\ & & & (U_a+Z_a)^{-1} & \\ & & & & (U_b+Z_b)^{-1} \end{pmatrix} \tag{5.1}$$

we will even show that $S(w)DF_\mu(w) \in \mathcal{L}(W_t', W_t)$, $t \in [p,q]$, is boundedly invertible with

$$\|(S(w)DF_\mu(w))^{-1}\|_{W_t',W_t} \leq C.$$

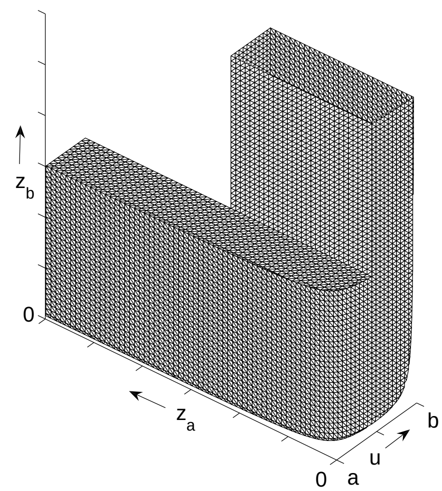

FIG. 5.1. *Sketch of the pointwise (u, z_a, z_b)-neighborhood N_ξ.*

Setting

$$\hat{U}_a = (U_a + Z_a)^{-1} U_a, \quad \hat{U}_b = (U_b + Z_b)^{-1} U_b, \\ \hat{Z}_a = (U_a + Z_a)^{-1} Z_a, \quad \hat{Z}_b = (U_b + Z_b)^{-1} Z_b, \tag{5.2}$$

we have $\hat{U}_a + \hat{Z}_a = I$, $\hat{U}_b + \hat{Z}_b = I$ and

$$S(w)DF_\mu(w) = \begin{pmatrix} \ell_{yy} & \ell_{yu} & c_y^* & 0 & 0 \\ \ell_{uy} & \ell_{uu} & c_u^* & -I & I \\ c_y & c_u & 0 & 0 & 0 \\ 0 & \hat{Z}_a & 0 & \hat{U}_a & 0 \\ 0 & -\hat{Z}_b & 0 & 0 & \hat{U}_b \end{pmatrix}$$

where we omit the arguments. For convenience, we use also the abbreviations

$$\hat{u}_a = \frac{u_a}{u_a + z_a}, \quad \hat{z}_a = \frac{z_a}{u_a + z_a}, \quad \hat{u}_b = \frac{u_b}{u_b + z_b}, \quad \hat{z}_b = \frac{z_b}{u_b + z_b}.$$

We show first that $S(w)DF_\mu(w)$ has a bounded inverse. This fact will play an essential role in this paper.

LEMMA 5.1. *Let (A1)–(A5)$_q$ hold for some $q \in]2, \infty]$ and let $w = (y, u, \lambda, z_a, z_b) \in N_{-\infty,q}(\mu)$ for some $\gamma \in (0,1)$, $\mu_{-\infty} > 0$.*

If the reduced Hessian $\hat{H}(y, u, \lambda)$ in (3.1) satisfies

$$(v, \hat{H}(y, u, \lambda)v) \geq \alpha \|v\|_2^2 \quad \forall\, v \in L^2(\Omega) \tag{5.3}$$

with some $\alpha > 0$ then $S(w)DF_\mu(w)$ has a bounded inverse in $\mathcal{L}(W_t', W_t)$ for all $t \in [p, q]$ and

$$\|(S(w)DF_\mu(w))^{-1}\|_{W_t', W_t} \leq C$$

with a constant $C > 0$. The constant C can be chosen uniformly on bounded subsets of $\{(\mu, w) \in (0, \infty) \times N_{-\infty,q}(\mu)\}$ on which (5.3) holds uniformly.

Moreover, $DF_\mu(w)$ has a bounded right inverse (in the case $q=\infty$ even a bounded inverse) $DF_\mu(w)^{-1}\in\mathcal{L}(W_t',W_t)$, $t\in[p,q]$, with

$$\|DF_\mu(w)^{-1}\|_{W_t',W_t}\le\frac{C'}{\min(1,\sqrt{\mu})},$$

where $C'=\frac{C}{\min(1,2\sqrt{\gamma})}$ with the above constant C.

Proof. We consider the equation

$$\begin{pmatrix}\ell_{yy} & \ell_{yu} & c_y^* & 0 & 0\\ \ell_{uy} & \ell_{uu} & c_u^* & -I & I\\ c_y & c_u & 0 & 0 & 0\\ 0 & \hat{Z}_a & 0 & \hat{U}_a & 0\\ 0 & -\hat{Z}_b & 0 & 0 & \hat{U}_b\end{pmatrix}\begin{pmatrix}s_y\\ s_u\\ s_\lambda\\ s_a\\ s_b\end{pmatrix}=\begin{pmatrix}r_y\\ r_u\\ r_\lambda\\ r_a\\ r_b\end{pmatrix}\tag{5.4}$$

with $r=(r_y,r_u,r_\lambda,r_a,r_b)\in W_t'$, $t\in[p,q]$, where we omit the arguments.

We note that by (A5)$_q$ the operator $S(w)DF_\mu(w)$ on the left hand side of (5.4) is in $\mathcal{L}(W_t,W_t')$, $p\le t\le q$.

Elimination with the last two block rows and subsequently with the first and third block row yields as above with $\hat{H}(y,u,\lambda)$ in (3.1) the reduced system

$$\begin{aligned}&(\hat{H}(y,u,\lambda)+\hat{U}_a^{-1}\hat{Z}_a+\hat{U}_b^{-1}\hat{Z}_b)s_u=\\ &=r_u+\hat{U}_a^{-1}r_a-\hat{U}_b^{-1}r_b+B_1r_\lambda-B_2r_y=:r_u'.\end{aligned}\tag{5.5}$$

with the abbreviations

$$B_1=c_u^*c_y^{-*}\ell_{yy}c_y^{-1}-\ell_{uy}c_y^{-1},\quad B_2=c_u^*c_y^{-*}.$$

Note that $B_1\in\mathcal{L}(\Lambda,L^q(\Omega))$, $B_2\in\mathcal{L}(V,L^q(\Omega))$ by (A5)$_q$.

Let

$$0<\varepsilon\le 1/2$$

and define $I_1=\{\hat{u}_a\le\varepsilon,\hat{u}_b>\varepsilon\}$, $I_2=\{\hat{u}_b\le\varepsilon,\hat{u}_a>\varepsilon\}$, $I_3=\{\hat{u}_a\le\varepsilon,\hat{u}_b\le\varepsilon\}$ and $I_4=\Omega\setminus(I_1\cup I_2\cup I_3)$. Multiply (5.5) by

$$D=\varepsilon I|_{I_4}+\hat{U}_a|_{I_1}+\hat{U}_b|_{I_2}+\min(\hat{U}_a,\hat{U}_b)|_{I_3},$$

where $I|_{I_4}=1_{I_4}I$, $\hat{U}_a|_{I_1}=\hat{u}_a1_{I_1}$, ….

Then we obtain on I_1

$$(\hat{U}_a|_{I_1}\hat{H}(y,u,\lambda)+(\hat{Z}_a+\hat{U}_a\hat{U}_b^{-1}\hat{Z}_b)|_{I_1})s_u=(\hat{U}_ar_u+r_a-\hat{U}_a\hat{U}_b^{-1}r_b+\hat{U}_aB_1r_\lambda-\hat{U}_aB_2r_y)|_{I_1}.$$

We have

$$0\le\hat{u}_a\hat{u}_b^{-1}|_{I_1}\le 1,\quad \hat{z}_a|_{I_1}\ge 1/2.$$

Thus, the right hand side is pointwise $\le|r_u|+|r_a|+|r_b|+|B_1r_\lambda|+|B_2r_y|$ and the operator on the left has the form $(\delta|_{I_1}I+D|_{I_1}\hat{H})$ with a function $\delta\in L^\infty(\Omega)$, $\delta\ge 1/2$.

On I_2 the situation is analogous. Similarly, we have on I_3 with $\hat{U}_{ab}:=\min(\hat{u}_a,\hat{u}_b)\cdot I$

$$\begin{aligned}&(\hat{U}_{ab}|_{I_3}\hat{H}(y,u,\lambda)+(\hat{U}_{ab}\hat{U}_a^{-1}\hat{Z}_a+\hat{U}_{ab}\hat{U}_b^{-1}\hat{Z}_b)|_{I_3})s_u\\ &=(\hat{U}_{ab}r_u+\hat{U}_{ab}\hat{U}_a^{-1}r_a-\hat{U}_{ab}\hat{U}_b^{-1}r_b+\hat{U}_{ab}B_1r_\lambda-\hat{U}_{ab}B_2r_y)|_{I_3}.\end{aligned}$$

Again, the right hand side is pointwise $\le |r_u| + |r_a| + |r_b| + |B_1 r_\lambda| + |B_2 r_y|$ and the operator on the left has the form $\delta|_{I_3} I + D|_{I_3}\hat{H})$ with $\delta \in L^\infty(\Omega)$, $\delta \ge 1/2$.

On I_4 we obtain

$$(\varepsilon I|_{I_4}\hat{H}(y,u,\lambda) + \varepsilon(\hat{U}_a^{-1}\hat{Z}_a + \hat{U}_b^{-1}\hat{Z}_b)|_{I_4})s_u = \varepsilon(r_u + \hat{U}_a^{-1}r_a - \hat{U}_b^{-1}r_b + B_1 r_\lambda - B_2 r_y)|_{I_4}.$$

Since $\hat{u}_a|_{I_4} > \varepsilon$ and $\hat{u}_b|_{I_4} > \varepsilon$ the right hand side is $\le \varepsilon|r_u| + |r_a| + |r_b| + \varepsilon|B_1 r_\lambda| + \varepsilon|B_2 r_y|$. The operator has on I_4 the form $\delta_{I_4} I + \varepsilon I_{I_4}\hat{H}$ with $\delta \ge 0$.

Thus, after multiplication with D the operator on the left hand side has the form $\delta I + D\hat{H}$ with $\delta|_{I_1\cup I_2\cup I_3} \ge 1/2$ and $\delta|_{I_4} \ge 0$ and the right hand side is pointwise $\le |r_u| + |r_a| + |r_b| + |B_1 r_\lambda| + |B_2 r_y|$. Moreover, we have

$$\| |r_u| + |r_a| + |r_b| + |B_1 r_\lambda| + |B_2 r_y| \|_t \le (3 + \|B_1\|_{\Lambda,L^t} + \|B_2\|_{\Sigma,L^t})\|r\|_{W_t'}. \tag{5.6}$$

Let without restriction $\alpha \le 1/2$ in (5.3). Then we have for all $s \in L^2(\Omega)$ with the abbreviations $s_i = s|_{I_i}$, $i = 1,\dots,4$,

$$\begin{aligned}(s,(\delta I + D\hat{H})s) \ge{}& \alpha(s_1,s_1) + \alpha(s_2,s_2) + \alpha(s_3,s_3) + \varepsilon\alpha(s_4,s_4)\\ &+ \varepsilon(s_4,\hat{H}(s_1+s_2+s_3)) + (s_1,\hat{U}_a\hat{H}s) + (s_2,\hat{U}_b\hat{H}s)\\ &+ (s_3,\hat{U}_{ab}\hat{H}s).\end{aligned}$$

Using that $(\rho u - v/\rho, \rho u - v/\rho) \ge 0$ for any $\rho > 0$ and thus

$$2(u,v) \le \rho^2(u,u) + \frac{1}{\rho^2}(v,v)$$

we obtain

$$(s_1,\hat{U}_a\hat{H}s) \ge -\frac{\alpha}{4}\|s_1\|_2^2 - \frac{1}{\alpha}\|\hat{U}_a|_{I_1}\hat{H}s\|_2^2 \ge -\frac{\alpha}{4}\|s_1\|_2^2 - \frac{\varepsilon^2}{\alpha}\|\hat{H}s\|_2^2$$

and analogously

$$\begin{aligned}(s_2,\hat{U}_b\hat{H}s) &\ge -\frac{\alpha}{4}\|s_2\|_2^2 - \frac{\varepsilon^2}{\alpha}\|\hat{H}s\|_2^2,\\ (s_3,\min(\hat{U}_a,\hat{U}_b)\hat{H}s) &\ge -\frac{\alpha}{4}\|s_3\|_2^2 - \frac{\varepsilon^2}{\alpha}\|\hat{H}s\|_2^2.\end{aligned}$$

Finally,

$$\varepsilon(s_4,\hat{H}(s_1+s_2+s_3)) \ge -\frac{\varepsilon\alpha}{4}\|s_4\|_2^2 - \frac{\varepsilon}{\alpha}\|\hat{H}(s_1+s_2+s_3)\|_2^2.$$

Now set

$$\varepsilon = \min(\alpha,\alpha^2)/(1 + 16\|\hat{H}\|_{L^2,L^2}^2).$$

Since $\|s\|_2^2 = \|s_1\|_2^2 + \|s_2\|_2^2 + \|s_3\|_2^2 + \|s_4\|_2^2$, inserting these estimates yields

$$(s,(\delta I + D\hat{H})s) \ge \frac{\alpha}{2}(s_1,s_1) + \frac{\alpha}{2}(s_2,s_2) + \frac{\alpha}{2}(s_3,s_3) + \frac{\varepsilon\alpha}{2}(s_4,s_4).$$

This shows together with (5.6) that

$$\|s_u\|_2 \le \frac{C}{\varepsilon\alpha}\|r\|_{W_2'} \tag{5.7}$$

where C depends only on α and $\hat{H}$ but not on μ.

To obtain a bound in L^t-topology we multiply (5.5) by s_u. By the structure of $\hat{H}$ according to $(\mathrm{A5})_q$ this yields the pointwise estimate

$$\begin{aligned} 0 &\le (\alpha_0 + \hat{u}_a^{-1}\hat{z}_a + \hat{u}_b^{-1}\hat{z}_b)s_u^2 \le s_u(r_u - \hat{H}_s s_u) + s_u\hat{u}_a^{-1}r_a - s_u\hat{u}_b^{-1}r_b \\ &+ s_u(B_1 r_\lambda) - s_u(B_2 r_y). \end{aligned} \tag{5.8}$$

Since $\hat{u}_a + \hat{z}_a = \hat{u}_b + \hat{z}_b = 1$, we have

$$\frac{\hat{u}_a^{-1}}{\alpha_0 + \hat{u}_a^{-1}\hat{z}_a + \hat{u}_b^{-1}\hat{z}_b} \le \frac{\hat{u}_a^{-1}}{\alpha_0 + \hat{u}_a^{-1}\hat{z}_a} = \frac{1}{\alpha_0\hat{u}_a + \hat{z}_a} \le \frac{1}{\min(\alpha_0, 1)}$$

and analogously

$$\frac{\hat{u}_b^{-1}}{\alpha_0 + \hat{u}_a^{-1}\hat{z}_a + \hat{u}_b^{-1}\hat{z}_b} = \frac{1}{\alpha_0\hat{u}_b + \hat{z}_b} \le \frac{1}{\min(\alpha_0, 1)}.$$

Division of (5.8) by $\left(\alpha_0 + \hat{u}_a^{-1}\hat{z}_a + \hat{u}_b^{-1}\hat{z}_b\right)|s_u|$ yields

$$|s_u| \le \frac{1}{\alpha_0}|r_u - \hat{H}_s s_u + B_1 r_\lambda - B_2 r_y| + \frac{1}{\min(\alpha_0, 1)}(|r_a| + |r_b|).$$

This yields together with (5.7) for all $t \in [p, q]$

$$\begin{aligned} \|s_u\|_t &\le \frac{1}{\alpha_0}\left(\|r_u\|_t + \|\hat{H}_s\|_{L^2,L^t}\|s_u\|_2 + \|B_1\|_{\Lambda,L^t}\|r_\lambda\|_\Lambda + \|B_2\|_{V,L^t}\|r_y\|_V\right) \\ &+ \frac{1}{\min(\alpha_0, 1)}\left(\|r_a\|_t + \|r_b\|_t\right) \\ &\le C'\|r\|_{W_t'}. \end{aligned} \tag{5.9}$$

We derive now also bounds for s_y, s_λ, s_a, s_b. We have

$$s_y = c_y^{-1}(-c_u s_u + r_\lambda)$$

and thus $L^t \subset L^p$ for $t \in [p, q]$ yields

$$\|s_y\|_Y \le \|c_y^{-1}c_u\|_{L^t,Y}\|s_u\|_t + \|c_y^{-1}\|_{\Lambda,Y}\|r_\lambda\|_\Lambda. \tag{5.10}$$

Next, we obtain

$$s_\lambda = c_y^{-*}(r_y - \ell_{yy}s_y - \ell_{yu}s_u),$$

which yields by $(\mathrm{A5})_q$

$$\|s_\lambda\|_\Sigma \le \|c_y^{-*}\|_{V,\Sigma}(\|r_y\|_V + \|\ell_{yy}\|_{Y,V}\|s_y\|_Y + \|\ell_{yu}\|_{L^t,V}\|s_u\|_t). \tag{5.11}$$

To estimate s_a, s_b we partition Ω into the sets

$$\Omega_a = \{u > (b+a)/2\} \cup \left\{u = (b+a)/2,\;\; z_a \le \frac{2\max(\mu_{-\infty}, \mu/\gamma)}{\nu}\right\}, \quad \Omega_a^c = \Omega \setminus \Omega_a.$$

Now (5.4) yields

$$\begin{aligned} s_a|_{\Omega_a} &= \hat{U}_a^{-1}(r_a - \hat{Z}_a s_u)|_{\Omega_a}, \\ s_b|_{\Omega_a^c} &= \hat{U}_b^{-1}(r_b + \hat{Z}_b s_u)|_{\Omega_a^c}, \\ s_a|_{\Omega_a^c} &= (\ell_{uy}s_y + \ell_{uu}s_u + c_u^* s_\lambda + s_b - r_u)|_{\Omega_a^c}, \\ s_b|_{\Omega_a} &= (-\ell_{uy}s_y - \ell_{uu}s_u - c_u^* s_\lambda + s_a + r_u)|_{\Omega_a}. \end{aligned}$$

By the definition of the neighborhood $N_{-\infty,q}(\mu)$ we have

$$z_a|_{\Omega_a} \le \frac{2\max(\mu_{-\infty},\mu/\gamma)}{\nu} =: C_\mu, \quad z_b|_{\Omega_a^c} \le \frac{2\max(\mu_{-\infty},\mu/\gamma)}{\nu} = C_\mu \tag{5.12}$$

and thus

$$\hat{u}_a|_{\Omega_a} = \left.\frac{u_a}{u_a+z_a}\right|_{\Omega_a} \ge \frac{1}{1+2C_\mu\nu^{-1}}, \quad \hat{u}_b|_{\Omega_a^c} = \left.\frac{u_b}{u_b+z_b}\right|_{\Omega_a^c} \ge \frac{1}{1+2C_\mu\nu^{-1}}.$$

Using that $0 \le \hat{z}_a \le 1$, $0 \le \hat{z}_b \le 1$ this yields for all $t \in [p,q]$

$$\begin{aligned}
\|s_a|_{\Omega_a}\|_t &\le (1+2C_\mu\nu^{-1})(\|r_a\|_t + \|s_u\|_t) \\
\|s_b|_{\Omega_a^c}\|_t &\le (1+2C_\mu\nu^{-1})(\|r_b\|_t + \|s_u\|_t) \\
\|s_a|_{\Omega_a^c}\|_t &\le \|l_{uy}\|_{Y,L^t}\|s_y\|_Y + \|l_{uu}\|_{L^t,L^t}\|s_u\|_t \\
&\quad + \|c_u^*\|_{\Sigma,L^t}\|s_\lambda\|_\Sigma + \|s_b|_{\Omega_a^c}\|_t + \|r_u\|_t \\
\|s_b|_{\Omega_a}\|_t &\le \|l_{uy}\|_{Y,L^t}\|s_y\|_Y + \|l_{uu}\|_{L^t,L^t}\|s_u\|_t \\
&\quad + \|c_u^*\|_{\Sigma,L^t}\|s_\lambda\|_\Sigma + \|s_a|_{\Omega_a}\|_t + \|r_u\|_t.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\|s_a\|_t &\le (1+2C_\mu\nu^{-1})(\|r_a\|_t + \|s_u\|_t) \\
&\quad + C'(\|s_y\|_Y + \|s_u\|_t + \|s_\lambda\|_\Sigma + \|r_u\|_t),
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
\|s_b\|_t &\le (1+2C_\mu\nu^{-1})(\|r_b\|_t + \|s_u\|_t) \\
&\quad + C'(\|s_y\|_Y + \|s_u\|_t + \|s_\lambda\|_\Sigma + \|r_u\|_t).
\end{aligned} \tag{5.14}$$

Now (5.9), (5.10), (5.11), (5.13), (5.14) yield

$$\|(S(w)DF_\mu(w))^{-1}\|_{W_t',W_t} \le C''.$$

It is easy to check that C'' can be chosen uniformly on bounded subsets of $\{(\mu,w) \in (0,\infty) \times N_{-\infty,q}(\mu)\}$ on which (5.3) holds uniformly.

Finally, the definition of the neighborhood $N_{-\infty,q}(\mu)$ yields

$$u_a + z_a \ge 2\sqrt{u_a z_a} \ge 2\sqrt{\gamma\mu}, \quad u_b + z_b \ge 2\sqrt{u_b z_b} \ge 2\sqrt{\gamma\mu}.$$

Therefore, the scaling matrix $S(w)$ in (5.1) satisfies

$$\|S(w)\|_{W_t',W_t'} \le \frac{1}{\min(1, 2\sqrt{\gamma\mu})}$$

and is invertible. Thus, $DF_\mu(w)^{-1} = (S(w)DF_\mu(w))^{-1}S(w)$ and

$$\begin{aligned}
\|DF_\mu(w)^{-1}\|_{W_t',W_t} &= \|(S(w)DF_\mu(w))^{-1}S(w)\|_{W_t',W_t} \\
&\le \|(S(w)DF_\mu(w))^{-1}\|_{W_t',W_t}\|S(w)\|_{W_t',W_t'} \le \frac{C''}{\min(1,2\sqrt{\gamma\mu})}.
\end{aligned}$$

□

We have the following variant of Lemma 5.1 that will be useful for showing the Hölder-continuity of the central path.

LEMMA 5.2. *Let the assumptions of Lemma 5.1 hold, but assume only that*

$$w = (y, u, \lambda, z_a, z_b) \in \frac{1}{2}(N_{-\infty,q}(\mu_1) + N_{-\infty,q}(\mu_2)) \tag{5.15}$$

with $\mu_1, \mu_2 \in (0, \infty)$ *and set* $\mu = \min(\mu_1, \mu_2)$.

Then $DF_\mu(w)$ *has a bounded right inverse* $DF_\mu(w)^{-1} \in \mathcal{L}(W_t', W_t)$, $t \in [p, q]$, *with*

$$\|DF_\mu(w)^{-1}\|_{W_t', W_t} \le \frac{C}{\min(1, \sqrt{\mu})} \tag{5.16}$$

where $C > 0$ *is a constant. The constant* C *can be chosen uniformly on bounded subsets of*

$$\left\{(\mu_1, \mu_2, w) \in (0, \infty)^2 \times \frac{1}{2}(N_{-\infty,q}(\mu_1) + N_{-\infty,q}(\mu_2))\right\}$$

on which (5.3) holds uniformly.

REMARK 5.3. (5.15) is weaker than $(y, u, \lambda, z_a, z_b) \in N_{-\infty,q}(\mu)$, since the constraints

$$z_a|_{\{u>(b+a)/2\}} \le \frac{2\max(\mu_{-\infty}, \mu/\gamma)}{\nu}, \quad z_b|_{\{u<(b+a)/2\}} \le \frac{2\max(\mu_{-\infty}, \mu/\gamma)}{\nu},$$
$$\min(z_a|_{\{u=(b+a)/2\}}, z_b|_{\{u=(b+a)/2\}}) \le \frac{2\max(\mu_{-\infty}, \mu/\gamma)}{\nu}$$

are nonconvex and can be violated by points satisfying (5.15).□

Proof. Let $\tilde{w} = (\tilde{y}, \tilde{u}, \tilde{\lambda}, \tilde{z}_a, \tilde{z}_b) \in N_{-\infty,q}(\mu_1)$, $\bar{w} = (\bar{y}, \bar{u}, \bar{\lambda}, \bar{z}_a, \bar{z}_b) \in N_{-\infty,q}(\mu_2)$, and $w = (y, u, \lambda, z_a, z_b) = \frac{1}{2}(\tilde{w} + \bar{w})$ according to (5.15). Without restriction we assume that $\mu_1 \le \mu_2$ then $\mu = \min(\mu_1, \mu_2) = \mu_1$.

We modify the proof of Lemma 5.1, but consider this time the system

$$DF_\mu(w)s = \hat{r} =: \begin{pmatrix} r_y \\ r_u \\ r_\lambda \\ \hat{r}_a \\ \hat{r}_b \end{pmatrix}, \text{ i.e., } S(w)DF_\mu(w)s = \begin{pmatrix} r_y \\ r_u \\ r_\lambda \\ (U_a + Z_a)^{-1}\hat{r}_a \\ (U_b + Z_b)^{-1}\hat{r}_b \end{pmatrix} =: \begin{pmatrix} r_y \\ r_u \\ r_\lambda \\ r_a \\ r_b \end{pmatrix},$$

which yields

$$\begin{pmatrix} \ell_{yy} & \ell_{yu} & c_y^* & 0 & 0 \\ \ell_{uy} & \ell_{uu} & c_u^* & -I & I \\ c_y & c_u & 0 & 0 & 0 \\ 0 & \hat{Z}_a & 0 & \hat{U}_a & 0 \\ 0 & -\hat{Z}_b & 0 & 0 & \hat{U}_b \end{pmatrix} \begin{pmatrix} s_y \\ s_u \\ s_\lambda \\ s_a \\ s_b \end{pmatrix} = \begin{pmatrix} r_y \\ r_u \\ r_\lambda \\ (U_a + Z_a)^{-1}\hat{r}_a \\ (U_b + Z_b)^{-1}\hat{r}_b \end{pmatrix} = \begin{pmatrix} r_y \\ r_u \\ r_\lambda \\ r_a \\ r_b \end{pmatrix}, \tag{5.17}$$

with $\hat{U}_a, \hat{U}_b, \hat{Z}_a, \hat{Z}_b$ in (5.2).

$(y, u, \lambda, z_a, z_b)$ according to (5.15) satisfies all constraints of $N_{-\infty,q}(\mu)$ with the possible exception of the nonconvex constraints

$$\begin{aligned} &z_a|_{\{u>(b+a)/2\}} \le \frac{2\max(\mu_{-\infty}, \mu/\gamma)}{\nu}, \quad z_b|_{\{u<(b+a)/2\}} \le \frac{2\max(\mu_{-\infty}, \mu/\gamma)}{\nu}, \\ &\min(z_a|_{\{u=(b+a)/2\}}, z_b|_{\{u=(b+a)/2\}}) \le \frac{2\max(\mu_{-\infty}, \mu/\gamma)}{\nu}. \end{aligned} \tag{5.18}$$

The only point, where the latter property is used in the proof of Lemma 5.1, is (5.12) for the derivation of (5.13), (5.14). Therefore, we still obtain the estimates (5.7), (5.9), (5.10), (5.11), which yield a constant $C > 0$ with

$$\|s_u\|_t + \|s_y\|_Y + \|s_\lambda\|_\Sigma \le C\|r\|_{W_t'} \quad \forall\, t \in [p, q]. \tag{5.19}$$

We derive now bounds for $\|s_a\|_t$ and $\|s_b\|_t$. Since (5.18) does not necessarily hold, we have to modify the proof of Lemma 5.1. Consider the subsets

$$\Omega_a = \left(\{\tilde{u} > (b+a)/2\} \cup \left\{\tilde{u} = (b+a)/2,\ \ \tilde{z}_a \le \frac{2\max(\mu_{-\infty}, \mu_1/\gamma)}{\nu}\right\}\right)$$
$$\cap \left(\{\bar{u} > (b+a)/2\} \cup \left\{\bar{u} = (b+a)/2,\ \ \bar{z}_a \le \frac{2\max(\mu_{-\infty}, \mu_2/\gamma)}{\nu}\right\}\right),$$
$$\Omega_b = \left(\{\tilde{u} < (b+a)/2\} \cup \left\{\tilde{u} = (b+a)/2,\ \ \tilde{z}_a > \frac{2\max(\mu_{-\infty}, \mu_1/\gamma)}{\nu}\right\}\right)$$
$$\cap \left(\{\bar{u} < (b+a)/2\} \cup \left\{\bar{u} = (b+a)/2,\ \ \bar{z}_a > \frac{2\max(\mu_{-\infty}, \mu_2/\gamma)}{\nu}\right\}\right).$$

This yields by using $\mu_1 \le \mu_2$

$$u_a|_{\Omega_a} \ge \frac{b-a}{2}, \quad z_a|_{\Omega_a} \le \frac{1}{2}\left(\frac{2\max(\mu_{-\infty}, \frac{\mu_1}{\gamma})}{\nu} + \frac{2\max(\mu_{-\infty}, \frac{\mu_2}{\gamma})}{\nu}\right) \le \frac{2\max(\mu_{-\infty}, \frac{\mu_2}{\gamma})}{\nu}$$

and similarly

$$u_b|_{\Omega_b} \ge \frac{b-a}{2}, \quad z_b|_{\Omega_b} \le \frac{1}{2}\left(\frac{2\max(\mu_{-\infty}, \frac{\mu_1}{\gamma})}{\nu} + \frac{2\max(\mu_{-\infty}, \frac{\mu_2}{\gamma})}{\nu}\right) \le \frac{2\max(\mu_{-\infty}, \frac{\mu_2}{\gamma})}{\nu}.$$

Hence, we have with $\Omega' = \Omega_a \cup \Omega_b$

$$z_a|_{\Omega_a} \le \frac{2\max(\mu_{-\infty}, \mu_2/\gamma)}{\nu}, \quad z_b|_{\Omega'\setminus\Omega_a} \le \frac{2\max(\mu_{-\infty}, \mu_2/\gamma)}{\nu}.$$

Thus, (5.12) holds on Ω' instead of Ω with μ_2 instead of μ and we obtain exactly as in the proof of Lemma 5.1 the following analogs of (5.13), (5.14) on Ω'

(5.20) $$\|s_a|_{\Omega'}\|_t \le (1 + 2C_{\mu_2}\nu^{-1})(\|r_a\|_t + \|s_u\|_t) + C'(\|s_y\|_Y + \|s_u\|_t + \|s_\lambda\|_\Sigma + \|r_u\|_t),$$

(5.21) $$\|s_b|_{\Omega'}\|_t \le (1 + 2C_{\mu_2}\nu^{-1})(\|r_b\|_t + \|s_u\|_t) + C'(\|s_y\|_Y + \|s_u\|_t + \|s_\lambda\|_\Sigma + \|r_u\|_t).$$

It remains to estimate $\|s_a|_{\Omega''}\|_t$, $\|s_b|_{\Omega''}\|_t$ for the set

$$\Omega'' = \Omega \setminus \Omega'.$$

By the definition of Ω' we have

$$u_a|_{\Omega''} = \left.\frac{\tilde{u}_a + \bar{u}_a}{2}\right|_{\Omega''} \ge \frac{b-a}{4} \ge \frac{\nu}{4}, \quad u_b|_{\Omega''} = \left.\frac{\tilde{u}_b + \bar{u}_b}{2}\right|_{\Omega''} \ge \frac{b-a}{4} \ge \frac{\nu}{4}.$$

We now split Ω'' into the sets

$$\Omega''_a = \left\{|\hat{Z}_a s_u| \le |r_a|\right\}, \quad \Omega''_b = \left\{|\hat{Z}_b s_u| \le |r_b|\right\}, \quad \Omega''_r = \Omega'' \setminus (\Omega''_a \cup \Omega''_b).$$

This yields by using (5.2)

$$|s_a|_{\Omega''_a}| = |\hat{U}_a^{-1}(r_a - \hat{Z}_a s_u)|_{\Omega''_a}| \le \left.\frac{2|u_a + z_a||r_a|}{u_a}\right|_{\Omega''_a} \le 8\nu^{-1}|\hat{r}_a|_{\Omega''_a}|,$$
$$|s_b|_{\Omega''_b}| = |\hat{U}_b^{-1}(r_b - \hat{Z}_b s_u)|_{\Omega''_b}| \le \left.\frac{2|u_b + z_b||r_b|}{u_b}\right|_{\Omega''_b} \le 8\nu^{-1}|\hat{r}_b|_{\Omega''_b}|,$$
$$s_a|_{\Omega''_b} = (\ell_{uy}s_y + \ell_{uu}s_u + c_u^* s_\lambda + s_b - r_u)|_{\Omega''_b},$$
$$s_b|_{\Omega''_a} = (-\ell_{uy}s_y - \ell_{uu}s_u - c_u^* s_\lambda + s_a + r_u)|_{\Omega''_a}.$$

Hence, we obtain on $\Omega_a'' \cup \Omega_b''$.

$$\|s_a|_{\Omega_a''\cup\Omega_b''}\|_t \le 8\nu^{-1}\|\hat{r}_a\|_t + C'(\|s_y\|_Y + \|s_u\|_t + \|s_\lambda\|_\Sigma + \|r_u\|_t), \tag{5.22}$$

$$\|s_b|_{\Omega_a''\cup\Omega_b''}\|_t \le 8\nu^{-1}\|\hat{r}_b\|_t + C'(\|s_y\|_Y + \|s_u\|_t + \|s_\lambda\|_\Sigma + \|r_u\|_t). \tag{5.23}$$

Finally, we have on Ω_r''

$$s_a|_{\Omega_r''} = \hat{U}_a^{-1}(r_a - \hat{Z}_a s_u)|_{\Omega_r''}, \quad s_b|_{\Omega_r''} = \hat{U}_b^{-1}(r_b + \hat{Z}_b s_u)|_{\Omega_r''}$$

and thus by the definition of Ω_r''

$$\begin{aligned} \operatorname{sgn}(s_a|_{\Omega_r''}) &= -\operatorname{sgn}(s_u|_{\Omega_r''}),\\ \operatorname{sgn}(s_b|_{\Omega_r''}) &= \operatorname{sgn}(s_u|_{\Omega_r''}). \end{aligned}$$

Hence, the second line in (5.17) yields

$$|s_a|_{\Omega_r''}| + |s_b|_{\Omega_r''}| = |(s_b - s_a)|_{\Omega_r''}| = |r_u - \ell_{uy}s_y - \ell_{uu}s_u - c_u^* s_\lambda|_{\Omega_r''}.$$

Therefore, (5.20), (5.21) hold also on Ω_r'' and we have shown that

$$\begin{aligned} \|s_a\|_t &\le 8\nu^{-1}\|\hat{r}_a\|_t + 2(1 + 2C_{\mu_2}\nu^{-1})(\|r_a\|_t + \|s_u\|_t)\\ &\quad + 2C'(\|s_y\|_Y + \|s_u\|_t + \|s_\lambda\|_\Sigma + \|r_u\|_t),\\ \|s_b\|_t &\le 8\nu^{-1}\|\hat{r}_b\|_t + 2(1 + 2C_{\mu_2}\nu^{-1})(\|r_b\|_t + \|s_u\|_t)\\ &\quad + 2C'(\|s_y\|_Y + \|s_u\|_t + \|s_\lambda\|_\Sigma + \|r_u\|_t). \end{aligned}$$

Together with (5.19) we conclude that the solution of (5.17) satisfies for all $t \in [p, q]$

$$\|s\|_{W_t} \le C''(\|r\|_{W_t'} + \|\hat{r}\|_{W_t}),$$

where C'' can be chosen uniformly for all $\mu_1, \mu_2 \in (0, \mu_0]$ only depending on μ_0.

Since $\mu = \min(\mu_1, \mu_2)$, we obtain as at the end of the proof of Lemma 5.1

$$u_a + z_a \ge 2\sqrt{\gamma\mu}, \quad u_b + z_b \ge 2\sqrt{\gamma\mu}$$

and thus by the definition of r in (5.17)

$$\|r\|_{W_t'} \le \max\left(1, \frac{1}{2\sqrt{\gamma\mu}}\right)\|\hat{r}\|_{W_t'}.$$

Therefore, (5.16) is proven, where the constant C can be chosen uniformly on bounded subsets of $\left\{(\mu_1, \mu_2, w) \in (0,\infty)^2 \times \frac{1}{2}(N_{-\infty,q}(\mu_1) + N_{-\infty,q}(\mu_2))\right\}$ on which (5.3) holds uniformly. ☐

6. Hölder continuity of the central path. We will now state conditions under which the central path defines a Hölder continuous curve that converges for $\mu \searrow 0$ to a solution of (2.1).

The analysis of the central path is quite obvious if $(A5)_q$ holds for $q = \infty$ and more involved in the case $q < \infty$. This is caused by the fact that

$$(u, z) \in (\mathcal{B}, \|\cdot\|_q) \times L^q \mapsto uz \in L^q$$

is only differentiable in the case $q = \infty$. Otherwise we have to weaken the image space to achieve differentiability. More precisely we have the following result.

LEMMA 6.1. *Let Z be an open bounded set in L^∞. Then for any $p < q \le \infty$ the mapping*

$$u, z \in (Z, \|\cdot\|_q) \times L^q \mapsto uz \in L^p$$

is continuously differentiable and the remainder term satisfies the estimate

$$\|(u+u')(z+z') - uz - (uz' + zu')\|_p = \|u'z'\|_p \le \|u'\|_{pq/(q-p)}\|z'\|_q.$$

If $p/(q-p) > 1$ *then the remainder term can further be estimated by*

$$\|u'\|_{pq/(q-p)}\|z'\|_q \le \|u'\|_q^{(q-p)/p}\|u'\|_\infty^{1-(q-p)/p}\|z'\|_q.$$

Moreover,

$$u, z \in L^\infty \times L^\infty \mapsto uz \in L^\infty$$

is continuously differentiable and the remainder term satisfies

$$\|(u+u')(z+z') - uz - (uz' + zu')\|_\infty = \|u'z'\|_\infty \le \|u'\|_\infty\|z'\|_\infty.$$

Proof. By the Hölder inequality the remainder term can be estimated as follows

$$\begin{aligned}\|(u+u')(z+z') - uz - (uz' + zu')\|_p = \|u'z'\|_p &\le \|(u')^p\|_{q/(q-p)}^{1/p}\|(z')^p\|_{q/p}^{1/p} \\ &= \|u'\|_{pq/(q-p)}\|z'\|_q.\end{aligned}$$

In particular, this shows that for any $p < q \le \infty$ the mapping $u, z \in (Z, \|\cdot\|_q) \times L^q \mapsto uz \in L^p$ is continuously differentiable. If $p/(q-p) > 1$ then interpolation between L^q and L^∞ yields

$$\|u'\|_{pq/(q-p)}\|z'\|_q \le \|u'\|_q^{(q-p)/p}\|u'\|_\infty^{1-(q-p)/p}\|z'\|_q.$$

Finally, $u, z \in L^\infty \times L^\infty \mapsto uz \in L^\infty$ is continuously differentiable, since the remainder term can be estimated by

$$\|(u+u')(z+z') - uz - (uz' + zu')\|_\infty = \|u'z'\|_\infty \le \|u'\|_\infty\|z'\|_\infty.$$

□

We consider now first the case that $q = \infty$.

LEMMA 6.2. *Let (A1)–(A4) and (A5)$_q$ with* $q = \infty$ *hold. If* $u \in \mathcal{B} \mapsto J(y(u), u)$ *is convex then for any* $\mu > 0$ *the central path* $\mu \in (0, \infty) \to w(\mu) \in W_\infty$ *according to (2.9) is well defined.*

If for $\mu_0 > 0$ *the reduced Hessian satisfies*

$$(v, \hat{H}(y(\mu), u(\mu), \lambda(\mu))v) \ge \alpha\|v\|_2^2 \quad \forall\, v \in L^2(\Omega),\ \ \forall\, \mu \in (0, \mu_0]$$

with some $\alpha > 0$ *then the central path* $\mu \in (0, \mu_0] \to w(\mu) \in W_\infty$ *is continuously differentiable, satisfies*

$$\|\dot{w}(\mu)\|_{W_\infty} \le \frac{C}{\sqrt{\mu}} \quad \forall\, \mu \in (0, \mu_0]$$

with a constant $C > 0$ *and is thus Hölder-continuous with index* $1/2$*. More precisely, we have with* $L = 2C$

$$\|w(\mu_1) - w(\mu_2)\|_{W_\infty} \le L|\sqrt{\mu_1} - \sqrt{\mu_2}| \le L\sqrt{|\mu_1 - \mu_2|} \quad \forall\, \mu_1, \mu_2 \in (0, \mu_0]. \tag{6.1}$$

Moreover, $\bar{w} = \lim_{\mu \searrow 0} w(\mu)$ *exists in* W_∞*,* $\bar{w} \in W_\infty$ *satisfies the KKT-conditions (2.9) and* $(\bar{y}, \bar{u})$ *is global solution of (2.1).*

Proof. By assumption the reduced objective function $u \in \mathcal{B} \to J(y(u), u)$ is convex. Since the barrier term ist strictly convex, the barrier problem (2.5) has therefore the strictly convex reduced objective function $u \mapsto J_\mu(y(u), u)$, and thus the solution $(y_\mu, u_\mu) = (y(u_\mu), u_\mu)$ provided by Proposition 2.3 is the unique solution of (2.5). Now also $\lambda_\mu, z_{a,\mu}, z_{b,\mu}$ are uniquely determined by the first and the last two equations in (2.9). Thus, together with Corollary 4.3 the central path $\mu \in (0, \infty) \to w(\mu) \in W_\infty$ is well defined and bounded on bounded subsets $(0, \mu_0]$.

The mapping $F_\mu : W_\infty \to W'_\infty$ is by (A1)–(A5)$_\infty$ and by Lemma 6.1 continuously differentiable. For $\mu > 0$ the primal-dual central path $\mu \mapsto w(\mu) := (y, u, \lambda, z_a, z_b)(\mu)$ given by (2.9) is the unique solution of

$$F_\mu((y, u, \lambda, z_a, z_b)(\mu)) = 0, \quad a \le u(\mu) \le b.$$

Since $w(\mu) \in N_{-\infty,\infty}(\mu)$, $DF_\mu(w(\mu)) \in \mathcal{L}(W_\infty, W'_\infty)$ has by Lemma 5.1 a bounded inverse. Thus, the implicit function theorem shows that $\mu \to w(\mu)$ is continuously differentiable and that the derivative w.r.t. μ satisfies

$$DF_\mu(w(\mu))\dot{w}(\mu) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}. \tag{6.2}$$

For fixed $\mu_0 > 0$ Lemma 5.1 yields a constant $C > 0$ with $\|DF_\mu(w(\mu))^{-1}\|_{W'_\infty, W_\infty} \le \frac{C}{\sqrt{\mu}}$ for all $\mu \in (0, \mu_0]$. With (6.2) we conclude that $\|\dot{w}(\mu)\|_{W_\infty} \le \frac{C}{\sqrt{\mu}}$ for all $\mu \in (0, \mu_0]$. This gives for all $0 < \mu_1 < \mu_2 \le \mu_0$

$$\begin{aligned} \|w(\mu_2) - w(\mu_1)\|_{W_\infty} &\le \int_{\mu_1}^{\mu_2} \|\dot{w}(\mu)\|_{W_\infty} \, d\mu \le \int_{\mu_1}^{\mu_2} \frac{C}{\sqrt{\mu}} \, d\mu = 2C(\sqrt{\mu_2} - \sqrt{\mu_1}) \\ &= 2C \frac{\mu_2 - \mu_1}{\sqrt{\mu_2} + \sqrt{\mu_1}} \le 2C\sqrt{\mu_2 - \mu_1}. \end{aligned}$$

This shows that $w(\cdot) \in C^{1/2}((0, \mu_0]; W_\infty)$ for any $\mu_0 > 0$. Hence, the central path is Hölder-continuous in W_∞ and admits a continuation until $\mu = 0$, i.e., $\bar{w} = \lim_{\mu \searrow 0} w(\mu)$ exists in W_∞. By continuity, $\bar{w}$ satisfies $F_0(\bar{w}) = 0$, which are just the KKT-conditions (2.4). Consequently, $(\bar{y}, \bar{u})$ is a global solution of (2.1), since the reduced objective functional and $\mathcal{B}$ are convex. ☐

For the general case we use the following auxiliary result.

LEMMA 6.3. *Let (A1)–(A4) and (A5)$_q$ with some $p < q < \infty$ hold and let $u \in \mathcal{B} \to J(y(u), u)$ be convex. Then for any $\mu > 0$ the central path $\mu \in (0, \infty) \to w(\mu) \in W_q$ according to (2.9) is well defined.*

If for $\mu_0 > 0$ the reduced Hessian satisfies

$$(v, \hat{H}(y(\mu), u(\mu), \lambda(\mu))v) \ge \alpha \|v\|_2^2 \quad \forall\, v \in L^2(\Omega), \ \forall\, \mu \in (0, \mu_0]$$

with some $\alpha > 0$ then the central path $\mu \in (0, \mu_0] \to w(\mu) \in W_q$ is continuous.

Proof. As at the beginning of the proof of Lemma 6.2 the barrier problem (2.5) has the strictly convex reduced objective function $u \mapsto J_\mu(y(u), u)$, and thus the solution $(\bar{y}, \bar{u}) = (y(\bar{u}), \bar{u})$ provided by Proposition 2.3 is the unique solution of (2.5). Now also $\bar{\lambda}, \bar{z}_a, \bar{z}_b$ are uniquely determined by the first and the last two equations in (2.9). Thus, together with Corollary 4.3 the central path $\mu \in (0, \infty) \to w(\mu) \in W_q$ is well defined and bounded on bounded subsets $(0, \mu_0]$.

We show next the continuity of the central path. Let $0 < \bar{\mu}$ be given and let $\bar{\mu} \le \mu < \eta$ be arbitrary. We write

$$J_\mu(y, u) = J(y, u) + \mu B(u)$$

with the log-barrier term B. We know that $B(u) \geq -c_B$ on $\mathcal{B}$ with some $c_B \geq 0$. Now we have for the solutions u_μ, u_η of the barrier problems (2.5)

$$\begin{aligned} J_\mu(y(u_\mu), u_\mu) &\leq J_\mu(y(u_\eta), u_\eta) = J_\eta(y(u_\eta), u_\eta) + (\mu - \eta)B(u_\eta) \\ &\leq J_\eta(y(u_\mu), u_\mu) + (\mu - \eta)B(u_\eta) \\ &= J_\mu(y(u_\mu), u_\mu) + (\mu - \eta)(B(u_\eta) - B(u_\mu)) \\ &\leq J_\mu(y(u_\mu), u_\mu) + (\eta - \mu)(c_B + B(u_\mu)). \end{aligned}$$

The first and third row shows that

$$B(u_\mu) - B(u_\eta) \geq 0$$

and thus in particular $B(u_\mu) \leq B(u_{\bar\mu})$. This yields

$$J_\mu(y(u_\mu), u_\mu) \leq J_\mu(y(u_\eta), u_\eta) \leq J_\mu(y(u_\mu), u_\mu) + (\eta - \mu)(c_B + B(u_{\bar\mu}))$$

and thus

$$|J_\mu(y(u_\eta), u_\eta) - J_\mu(y(u_\mu), u_\mu)| \leq (\eta - \mu)|c_B + B(u_{\bar\mu})| \to 0 \quad \text{for } \eta - \mu \to 0, \;\; \bar\mu \leq \mu < \eta.$$

As we have already observed, (A1)–(A4) imply that the reduced objective functional is twice continuously differentiable. Since the barrier terms are convex and $J'_\mu(y(u_\mu), u_\mu) = 0$, we have

$$J_\mu(y(u_\eta), u_\eta) - J_\mu(y(u_\mu), u_\mu) \geq (u_\eta - u_\mu, \hat H(u(\tau))(u_\eta - u_\mu)) \geq \alpha \|u_\eta - u_\mu\|_2^2$$

with $u(\tau) = u_\mu + \tau(u_\eta - u_\mu)$ and appropriate $\tau \in [0,1]$. This yields

$$u_\eta - u_\mu \to 0 \text{ in } L^2 \text{ for } \eta \to \mu, \;\; \eta, \mu \geq \bar\mu$$

and thus in all L^s, $s < \infty$ by interpolation with the uniform L^∞-bound $\|u_\eta - u_\mu\|_\infty \leq \|b - a\|_\infty$. Now Lemma 4.6 yields

$$(u_\eta, y_\eta, \lambda_\eta, z_{a,\eta}, z_{b,\eta}) \to (u_\mu, y_\mu, \lambda_\mu, z_{a,\mu}, z_{b,\mu}) \;\; \text{in } W_q \text{ for } \eta \to \mu, \;\; \eta, \mu \geq \bar\mu.$$

□

LEMMA 6.4. *Let the assumptions of Lemma 6.3 hold. Then in addition to Lemma 6.3 the central path $\mu \in (0, \mu_0] \to w(\mu) \in W_q$ satisfies*

$$\limsup_{\eta \to \mu} \frac{\|w(\eta) - w(\mu)\|_{W_q}}{\eta - \mu} \leq \frac{C}{\sqrt{\mu}} \quad \forall\, \mu \in (0, \mu_0] \tag{6.3}$$

with a constant $C > 0$ and is thus Hölder-continuous with index $1/2$ in all spaces W_t, $t \in [p, q]$. More precisely, we have with $L = 2C$

$$\|w(\mu_1) - w(\mu_2)\|_{W_t} \leq L|\sqrt{\mu_1} - \sqrt{\mu_2}| \leq L\sqrt{|\mu_1 - \mu_2|} \quad \forall\, \mu_1, \mu_2 \in (0, \mu_0]. \tag{6.4}$$

Moreover, $\bar w = \lim_{\mu \searrow 0} w(\mu)$ exists in W_q, $\bar w \in W_q$ satisfies the KKT-conditions (2.9) and $(\bar y, \bar u)$ is global solution of (2.1).

Proof. By assumptions (A1), (A5$_q$) the first three components of the mapping

$$F_\mu : W_q \to V \times L^q \times \Lambda \times L^{q/2} \times L^{q/2}$$

are continuously differentiable. Moreover, we have

$$u_a(\eta) z_a(\eta) - u_a(\mu) z_a(\mu) = \frac{u_a(\eta) + u_a(\mu)}{2}(z_a(\eta) - z_a(\mu)) + \frac{z_a(\eta) + z_a(\mu)}{2}(u_a(\eta) - u_a(\mu)).$$

Therefore, we obtain with

$$\tilde{w}(\mu) := \left(y(\mu), \frac{u(\eta)+u(\mu)}{2}, \lambda(\mu), \frac{z_a(\eta)+z_a(\mu)}{2}, \frac{z_b(\eta)+z_b(\mu)}{2} \right)$$

the identity

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta-\mu \\ \eta-\mu \end{pmatrix} = F_\mu(w(\eta)) - F_\mu(w(\mu)) \tag{6.5}$$

$$= DF_\mu(\tilde{w}(\mu))(w(\eta)-w(\mu)) + o_{W_q'}(\|w(\eta)-w(\mu)\|_{W_q}),$$

where

$$\frac{\|o_{W_q'}(\|w(\eta)-w(\mu)\|_{W_q})\|_{W_q'}}{\|w(\eta)-w(\mu)\|_{W_q}} \to 0 \quad \text{for } \eta \to \mu,$$

since $\|w(\eta)-w(\mu)\|_{W_q} \to 0$ for $\eta \to \mu$ by Lemma 6.3.

Since

$$\tilde{w}(\eta) \in \frac{1}{2}(N_{-\infty,q}(\mu) + N_{-\infty,q}(\eta)),$$

Lemma 5.2 yields for the right inverse

$$\|DF_\mu(\tilde{w}(\eta))^{-1}\|_{W_q'.W_q} \le \frac{C}{\sqrt{\min(\mu,\eta)}}.$$

For $|\eta-\mu| \le \varepsilon$, $\varepsilon > 0$ small enough we have with the remainder term in (6.5) clearly

$$\left\| w(\eta) - w(\mu) + DF_\mu(\tilde{w}(\eta))^{-1} o_{W_q'}(\|w(\eta)-w(\mu)\|_{W_q}) \right\|_{W_q} \ge \frac{1}{2}\|w(\eta)-w(\mu)\|_{W_q}$$

for all $|\eta-\mu| \le \varepsilon$. We conclude with (6.5) that

$$\frac{\|w(\eta)-w(\mu)\|_{W_q}}{\eta-\mu} \le \frac{2C}{\sqrt{\min(\mu,\eta)}} \left\| \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\|_{W_q'} \le \frac{2CC'}{\sqrt{\min(\mu,\eta)}} \quad \text{for all } |\eta-\mu| \le \varepsilon.$$

This shows (6.3) and the Hölder continuity with index $1/2$ follows immediately. By writing the integral $\int_{\mu_1}^{\mu_2} \frac{C}{\sqrt{\mu}}\, d\mu$ as a limit of Riemann sums and by using (6.3), we see that again

$$\|w(\mu_1)-w(\mu_2)\|_{W_t} \le \left| \int_{\mu_1}^{\mu_2} \frac{C}{\sqrt{\mu}}\, d\mu \right| = 2C|\sqrt{\mu_1} - \sqrt{\mu_2}|$$

$$\le 2C\sqrt{|\mu_1-\mu_2|} \quad \forall\, \mu_1, \mu_2 \in (0, \mu_0].$$

The fact that $\bar{w} = \lim_{\mu \searrow 0} w(\mu)$ exists in W_q, satisfies (2.4) and is global solution of (2.1) follows now exactly as at the end of the proof of Lemma 6.2. □

REMARK 6.5. *As usual for interior-point methods, we used a convexity assumption to ensure that the central path is a uniquely defined curve. Here and in the following, extensions to more general, non-convex, settings are possible by considering neighborhoods where the central path is a unique curve.*

7. A primal-dual interior-point method. Let (A1)–(A5)$_q$ hold. The previous considerations show that for $w \in N_{-\infty,q}(\mu)$ the solution s of the primal-dual Newton system

$$DF_\mu(w)s = -F_\mu(w)$$

is only contained in W_q. Therefore, in the case $q < \infty$ we cannot ensure $w + \alpha s \in N_{-\infty,q}(\mu)$ by choosing an appropriate stepsize $\alpha \in (0, 1]$. Instead, we use in addition a projection onto the neighborhood $N_{-\infty,q}(\mu)$.

DEFINITION 7.1. *We denote by P_μ a projection onto $N_{-\infty,q}(\mu)$ in W_q, i.e.*

$$P_\mu(w) \in N_{-\infty,q}(\mu), \quad \|P_\mu(w) - w\|_{W_q} = \min\left\{\|\tilde{w} - w\|_{W_q} \ : \ \tilde{w} \in N_{-\infty,q}(\mu)\right\}.$$

If more than one projection point exists, P_μ selects one of them.

Obviously,

$$P_\mu(y, u, \lambda, z_a, z_b) = (y, *, \lambda, *, *),$$

i.e. P_μ does not change the y- and λ-component. Furthermore, the projection does not depend on q, since it reduces to a pointwise projection in $\mathbb{R}^3$ with respect to the Euclidean norm of the (u, z_a, z_b)-part.

REMARK 7.2. The form of our neighborhood $N_{-\infty,q}$ allows for a pointwise computation of the projection P_μ. In fact, for almost all $\xi \in \Omega$ we have to project the point $(u(\xi), z_a(\xi), z_b(\xi)) \in \mathbb{R}^3$ onto the set (see the sketch in Figure 5.1)

$$\begin{aligned} N_\xi = &\left\{(v, s_a, s_b) \in \mathbb{R}^3 \ : \ s_a, s_b > 0, \ (v - a(\xi))s_a \ge \gamma\mu, \ (b(\xi) - v)s_b \ge \gamma\mu\right\} \\ &\cap \Big(([\frac{a(\xi) + b(\xi)}{2}, b(\xi)] \times [0, \frac{2\max(\mu_{-\infty}, \mu/\gamma)}{\nu}] \times [0, \infty)) \\ &\quad \cup ([a(\xi), \frac{a(\xi) + b(\xi)}{2}] \times [0, \infty) \times [0, \frac{2\max(\mu_{-\infty}, \mu/\gamma)}{\nu}]\Big). \end{aligned}$$

The first set is convex and the second set is the union of two cuboids of infinite length. There are several possibilities for computing the projection. One approach consists in projecting onto the boundary surfaces and (if necessary) onto the edges. The main work then consists in computing the projection onto the two surfaces generated by the hyperbolic constraints, and possibly onto their intersection. These three projections can be computed one after the other, and we can stop as soon as the point lies in N_ξ. They can be reduced to one dimensional least squares problems that can be solved with a Newton- or Gauss-Newton iteration of complexity $O(1)$. All other cases turn out to be easy. We implemented the projection along these lines. It is significantly less expensive than solving the primal-dual Newton system. □

We show now that P_μ has a Lipschitz constant ≤ 2.

LEMMA 7.3. *Let (A5)$_q$ hold and let $w_\mu = (y_\mu, u_\mu, \lambda_\mu, z_{a,\mu}, z_{b,\mu}) \in W_q$ be a point on the central path. Then we have*

$$\|P_\mu(w) - w_\mu\|_{W_q} \le 2\|w - w_\mu\|_{W_q} \quad \forall\, w \in W_q.$$

Proof. Since $w_\mu \in N_{-\infty,q}(\mu)$, we have

$$\|P_\mu(w) - w\|_{W_q} \le \|w_\mu - w\|_{W_q}.$$

Hence,

$$\|P_\mu(w) - w_\mu\|_{W_q} \le \|P_\mu(w) - w\|_{W_q} + \|w - w_\mu\|_{W_q} \le 2\|w - w_\mu\|_{W_q}.$$

□

We consider now the following conceptual algorithm.

Algorithm PDPF: Projected Primal-Dual Interior-Point Method.

1. Choose $\mu_0 > 0$, $\gamma \in (0,1)$, $\mu_{-\infty} > 0$, $\sigma_{\min} \in (0,1)$, $\mu_0 \in (0, \mu_{-\infty}]$, and
$$w_0 := (y_0, u_0, \lambda_0, z_{a,0}, z_{b,0}) \in N_{-\infty,q}(\mu_0).$$
Set $k := 0$.
2. Solve the primal-dual Newton system
$$DF_{\mu_k}(w_k)s_k = -F_{\mu_k}(w_k)$$
and set $w_{k+1} := P_{\mu_k}(w_k + s_k)$.
3. Choose $\sigma_k \in [\sigma_{\min}, 1]$ and set $\mu_{k+1} := \sigma_k \mu_k$.
4. Set $k := k+1$ and goto step 2.

REMARK 7.4. The algorithm is formulated here as compact as possible. The method can be augmented by additional features and safeguards. In particular, a stepsize can be introduced to damp the full step if necessary. The choice of the stepsize could be based on a decrease condition for a merit function, the simplest one being an appropriate norm of the residual F_{μ_k}. A detailed study of this issue would go beyond the purpose of this paper.

We will analyze Algorithm PDPF under assumption (A1)–(A5)$_\infty$. If merely (A1)–(A5)$_q$ for $q < \infty$ hold, we will have to modify the algorithm by introducing a smoothing step, see Algorithm PDPFS in section 9. Appropriate implementations of Algorithm PDPFS will even yield superlinear convergence. ☐

8. Global linear convergence for the L^∞-setting. We assume throughout this section that the assumptions (A1)–(A4) and (A5)$_\infty$ hold.

8.1. Quadratic local convergence towards the central path. We show first that the primal-dual iteration

$$DF_\mu(w)\, s = -F_\mu(w), \quad w_+ = w + s. \tag{8.1}$$

yields quadratic local convergence towards the central path.

LEMMA 8.1. *Let $\mu_0 > 0$ and $\rho_0 > 0$ be fixed. Assume that (A1)–(A4) and (A5)$_\infty$ hold, that $u \in \mathcal{B} \mapsto J(y(u), u)$ is convex (to ensure uniqueness of the central path), and that*

$$\begin{aligned} &(v, \hat{H}(y,u,\lambda)v) \geq \alpha \|v\|_2^2 \quad \forall\, v \in L^2(\Omega), \\ &\qquad\qquad\qquad\qquad \forall\, w \in N_{-\infty,\infty}(\mu),\ \|w - w(\mu)\|_{W_\infty} \leq \rho_0,\ \mu \in (0, \mu_0]. \end{aligned} \tag{8.2}$$

Then there exists a constant $C > 0$ such that for all $0 < \mu \leq \mu_0$ and for all $w \in N_{-\infty,\infty}(\mu)$ with $\|w - w(\mu)\|_\infty \leq \rho_0$ the solution w_+ of the primal dual Newton step (8.1) satisfies

$$\|w_+ - w(\mu)\|_{W_\infty} \leq \frac{C}{\sqrt{\mu}} \|w - w(\mu)\|_{W_\infty}^2.$$

For the projected iterate $P_\mu(w_+) \in N_{-\infty,\infty}(\mu)$ the estimate holds

$$\|P_\mu(w_+) - w(\mu)\|_{W_\infty} \leq \frac{2C}{\sqrt{\mu}} \|w - w(\mu)\|_{W_\infty}^2.$$

and thus the projected iteration converges locally with quadratic rate.

Proof. We know by Corollary 4.3 that $w(\mu)$ is uniformly bounded in W_∞ for all $\mu \in (0, \mu_0]$. Thus, $\|w - w(\mu)\|_{W_\infty} \leq \rho_0$ implies $\|w\|_{W_\infty} \leq M$ for some constant $M > 0$. Hence, all μ, w in (8.2) are contained in a bounded subset of $\{(\mu, w) \in (0,\infty) \times N_{-\infty,\infty}(\mu)\}$ and Lemma 5.1 yields a constant $C > 0$ with

$$\|DF_\mu(w)^{-1}\|_{W_\infty', W_\infty} \leq \frac{C}{\sqrt{\mu}}.$$

We have

$$
DF_\mu(w)(w_+ - w) = -F_\mu(w), \quad DF_\mu(w)(w(\mu) - w(\mu)) = -F_\mu(w(\mu)), \tag{8.3}
$$

and thus

$$
DF_\mu(w)(w_+ - w(\mu)) = F_\mu(w(\mu)) - F_\mu(w) - DF_\mu(w)(w(\mu) - w). \tag{8.4}
$$

Since the first three components of $F_\mu : W_\infty \to W'_\infty$ are by (A1), (A5)$_q$ Lipschitz continuously differentiable on bounded subsets, this gives

$$
DF_\mu(w)(w_+ - w(\mu)) = \begin{pmatrix} R_1(w(\mu) - w) \\ R_2(w(\mu) - w) \\ R_3(w(\mu) - w) \\ (u(\mu) - u)(z_a(\mu) - z_a) \\ (u(\mu) - u)(z_b(\mu) - z_b) \end{pmatrix}, \tag{8.5}
$$

where with a Lipschitz constant $L > 0$

$$
\|R_1(w(\mu) - w)\|_V + \|R_2(w(\mu) - w)\|_\infty + \|R_3(w(\mu) - w)\|_\Lambda \le L\|w(\mu) - w\|^2_{W_\infty}.
$$

Therefore,

$$
\begin{aligned}
\|w_+ - w(\mu)\|_{W_\infty} \le \|DF_\mu(w)^{-1}\|_{W'_\infty, W_\infty} \big(L\|w(\mu) - w\|^2_{W_\infty} \\
+ \|(u(\mu) - u)(z_a(\mu) - z_a)\|_\infty + \|(u(\mu) - u)(z_b(\mu) - z_b)\|_\infty\big).
\end{aligned}
$$

This yields

$$
\|w_+ - w(\mu)\|_{W_\infty} \le \frac{C}{\sqrt{\mu}}(2 + L)\|w - w(\mu)\|^2_{W_\infty}.
$$

Finally, the estimate for $P_\mu(w_+)$ follows from Lemma 7.3. ☐

8.2. Global linear convergence of the interior-point method. The previous result yields linear convergence for a short step method.

THEOREM 8.2. *Let $\mu_0 > 0$ and $\rho_0 > 0$ be fixed. Assume that (A1)–(A4), (A5)$_\infty$ hold, that $u \in \mathcal{B} \mapsto J(y(u), u)$ is convex (to ensure uniqueness of the central path), and that*

$$
\begin{aligned}
(v, \hat{H}(y, u, \lambda)v) \ge \alpha\|v\|_2^2 \quad & \forall\, v \in L^2(\Omega), \\
& \forall\, w \in N_{-\infty,\infty}(\mu),\ \|w - w(\mu)\|_{W_\infty} \le \rho_0,\ \mu \in (0, \mu_0].
\end{aligned} \tag{8.2}
$$

Then there are constants $\bar{\rho} \in (0, \rho_0]$ and $\bar{\sigma}_{\min} \in (0, 1)$ such that Algorithm PFPF has the following convergence property:

For any starting point $w_0 \in N_{-\infty,\infty}(\mu_0)$ with $\|w_0 - w(\mu_0)\|_{W_\infty} \le \bar{\rho}$, Algorithm PDPF with $\sigma_{\min} \in (\bar{\sigma}_{\min}, 1)$ generates a sequence $w_k \in N_{-\infty,\infty}(\mu_k)$ with

$$
\|w_k - w(\mu_k)\|_{W_\infty} \le C\sqrt{\mu_k} \tag{8.6}
$$

$$
\|w_k - \bar{w}\|_{W_\infty} \le (C + L)\sqrt{\mu_k} \tag{8.7}
$$

$$
\mu_k = \sigma_0 \cdots \sigma_{k-1}\mu_0. \tag{8.8}
$$

Here, $C > 0$ is a constant, $L > 0$ is the Hölder constant of the central path on $(0, \mu_0]$, and $\bar{w} = \lim_{\mu \searrow 0} w(\mu)$ is the solution of (2.1). In particular, if, for all k, $\sigma_k \in [\sigma_{\min}, \bar{\sigma}]$ is chosen with $\sigma_{\min} \le \bar{\sigma} < 1$, then (w_k) converges to $\bar{w}$ with R-linear rate.

Proof. Consider an arbitrary $\mu \in (0, \mu_0]$. Then there exists by Lemma 8.1 a constant $C > 0$ such that for any $w \in N_{-\infty,\infty}(\mu)$ with $\|w - w(\mu)\|_{W_\infty} \le \rho_0$ the estimate

$$\|P_\mu(w_+) - w(\mu)\|_{W_\infty} \le 2\|w_+ - w(\mu)\|^2_{W_\infty} \le \frac{2C}{\sqrt{\mu}}\|w - w(\mu)\|^2_{W_\infty}$$

holds, where w_+ is the result of the primal-dual Newton step (8.1).

Now fix $\tau \in (0, 1)$ such that

$$\bar{\rho} := \frac{\tau\sqrt{\mu_0}}{2C} \le \rho_0. \tag{8.9}$$

Then for any $w \in N_{-\infty,\infty}(\mu)$ with

$$\|w - w(\mu)\|_{W_\infty} \le \frac{\tau\sqrt{\mu}}{2C},$$

we have

$$\|P_\mu(w_+) - w(\mu)\|_{W_\infty} \le \frac{2C}{\sqrt{\mu}}\|w - w(\mu)\|^2_{W_\infty} \le \tau\|w - w(\mu)\|_{W_\infty} \le \frac{\tau^2\sqrt{\mu}}{2C}. \tag{8.10}$$

Moreover, we have with the Hölder constant L of the central path in (6.1) for $0 < \sigma < 1$

$$\|w(\mu) - w(\sigma\mu)\|_{W_\infty} \le L(1 - \sqrt{\sigma})\sqrt{\mu}.$$

and thus

$$\|P_\mu(w_+) - w(\sigma\mu)\|_{W_\infty} \le \frac{\tau^2\sqrt{\mu}}{2C} + L(1 - \sqrt{\sigma})\sqrt{\mu}.$$

Therefore, we can ensure that the new iterate satisfies

$$\|P_\mu(w_+) - w(\sigma\mu)\|_{W_\infty} \le \frac{\tau\sqrt{\sigma\mu}}{2C}, \tag{8.11}$$

if $\sigma \in (0, 1)$ is chosen such that

$$\frac{\tau^2}{2C} + L(1 - \sqrt{\sigma}) \le \frac{\tau\sqrt{\sigma}}{2C}.$$

Since $\tau \in (0, 1)$, this holds for $\sigma \in (0, 1)$ sufficiently close to 1, more precisely for

$$1 > \sigma \ge \bar{\sigma}_{\min} := \left(\frac{\tau^2 + 2LC}{\tau + 2LC}\right)^2. \tag{8.12}$$

Thus we obtain by induction: If $\bar{\rho}$ is chosen according to (8.9) and $\bar{\sigma}_{\min}$ is given by (8.12) then Algorithm PDPF with $\sigma_{\min} \in [\bar{\sigma}_{\min}, 1)$ generates a sequence (w_k) with $w_{k+1} = P_{\mu_k}(w_k + s_k)$ and (see (8.11))

$$\|w_{k+1} - w(\mu_{k+1})\|_{W_\infty} \le \frac{\tau\sqrt{\mu_{k+1}}}{2C} \le \bar{\rho}, \quad \mu_{k+1} = \sigma_k\mu_k, \tag{8.13}$$

and (see (8.10))

$$\|w_{k+1} - w(\mu_k)\|_{W_\infty} \le \frac{\tau^2\sqrt{\mu_k}}{2C} \le \bar{\rho}. \tag{8.14}$$

With the solution $\bar{w} = \lim_{\mu\searrow 0} w(\mu)$ of (2.1) we have by (8.13) in addition

$$\|w_k - \bar{w}\|_{W_\infty} \le \|w_k - w(\mu_k)\|_{W_\infty} + \|w(\mu_k) - \bar{w}\|_{W_\infty} \le \frac{\tau\sqrt{\mu_k}}{2C} + L\sqrt{\mu_k}.$$

This proves (8.6), (8.7), (8.8).

The R-linear convergence follows trivially if σ_k is bounded above by $\bar{\sigma} < 1$. ∎

9. Global linear and superlinear local convergence for the general L^q-setting. If $(A5)_q$ holds only for some $p < q < \infty$ the convergence analysis is more delicate. Under a strict complementarity assumption and by using an additional smoothing step we will prove global linear convergence in the general L^q-setting. Moreover, we will also show that superlinear local convergence is achieved if μ_k is reduced fast enough.

To keep the presentation compact, we exclude the case $q = \infty$, since this would have to be discussed separately at several places. Nevertheless, counterparts of the following results can be obtained also for the case $q = \infty$.

We refine our previous analysis as follows. Since under the assumptions of Lemma 5.1

$$\|DF_\mu(w(\mu))^{-1}\|_{W_t',W_t} = O(\mu^{-1/2}), \text{ but } \|(S(w)DF_\mu(w))^{-1}\|_{W_t',W_t} \le C, \quad t \in [p,q],$$

with the scaling operator

$$S(w) = \begin{pmatrix} I & & & & \\ & I & & & \\ & & I & & \\ & & & (U_a + Z_a)^{-1} & \\ & & & & (U_b + Z_b)^{-1} \end{pmatrix}, \tag{5.1}$$

we use now for the analysis of the primal-dual Newton iteration instead of (8.4) the scaled equation

$$S(w)DF_\mu(w)(w_+ - w(\mu)) = S(w)\left(F_\mu(w(\mu)) - F_\mu(w) - DF_\mu(w)(w(\mu) - w)\right). \tag{9.1}$$

Then we use a two-norm technique based on the $L^p - L^q$ norm gap to estimate the right hand side. Independently of the size of $\mu > 0$ this will yield an estimate of the form

$$\|w_+ - w(\mu)\|_{W_p} = o(\|w - w(\mu)\|_{W_q}).$$

The norm gap will then be closed by using a smoothing step. We recall that with the notations

$$\hat{U}_a = (U_a + Z_a)^{-1}U_a, \quad \hat{U}_b = (U_b + Z_b)^{-1}U_b,$$
$$\hat{Z}_a = (U_a + Z_a)^{-1}Z_a, \quad \hat{Z}_b = (U_b + Z_b)^{-1}Z_b,$$

we have $\hat{U}_a + \hat{Z}_a = I$, $\hat{U}_b + \hat{Z}_b = I$ and

$$S(w)DF_\mu(w) = \begin{pmatrix} \ell_{yy} & \ell_{yu} & c_y^* & 0 & 0 \\ \ell_{uy} & \ell_{uu} & c_u^* & -I & I \\ c_y & c_u & 0 & 0 & 0 \\ 0 & \hat{Z}_a & 0 & \hat{U}_a & 0 \\ 0 & -\hat{Z}_b & 0 & 0 & \hat{U}_b \end{pmatrix}$$

where we omit the arguments.

9.1. Refined analysis of the primal-dual Newton step. We first prove a similar result as in Lemma 8.1 where we avoid the μ-dependent convergence factor by using a two-norm technique. We will need a strict complementarity assumption.

DEFINITION 9.1. *Let $\bar{w} = (\bar{y}, \bar{u}, \bar{\lambda}, \bar{z}_a, \bar{z}_b) \in Y \times U \times \Lambda^* \times U^* \times U^*$ satisfy the KKT-conditions (2.4). Then* strict complementarity *holds at $\bar{w}$ if*

$$\text{leb}\left(\{\min(\bar{u}_a + \bar{z}_a, \bar{u}_b + \bar{z}_b) = 0\}\right) = 0.$$

where leb () *is the Lebesgue measure on Ω.*

We define the function

$$\omega(t) = \text{leb}\left(\{\min(\bar{u}_a + \bar{z}_a, \bar{u}_b + \bar{z}_b) \le t\}\right). \tag{9.2}$$

Under a strict complementarity assumption we then have

$$\lim_{t \searrow 0} \omega(t) = 0. \tag{9.3}$$

If $\omega(t) = O(t^\kappa)$ as $t \searrow 0$, we say that strong strict complementarity holds.

DEFINITION 9.2. *Let $\bar{w} = (\bar{y}, \bar{u}, \bar{\lambda}, \bar{z}_a, \bar{z}_b) \in Y \times U \times \Lambda^* \times U^* \times U^*$ satisfy the KKT-conditions (2.4). Then* strong strict complementarity *holds at $\bar{w}$ if there exist constants $C_c > 0$, $\kappa > 0$ such that*

$$\omega(t) \le C_c t^\kappa \quad \forall\, t \ge 0. \tag{9.4}$$

We start with the following technical result.

LEMMA 9.3. *Let the assumptions of Lemma 6.3 hold and let $\bar{w} = \lim_{\mu \searrow 0} w(\mu)$ be the solution of (2.4). Define the function*

$$\omega_\mu(t) = \text{leb}\left(\{\min(u_a(\mu) + z_a(\mu), u_b(\mu) + z_b(\mu)) \le t\}\right). \tag{9.5}$$

Then there exists a constant $C > 0$ with

$$\omega_\mu(t) \begin{cases} = 0 & \text{for } t < 2\sqrt{\mu}, \\ \le \omega(2\max(t, t^{\frac{q}{q+1}})) + C\max(t, t^{\frac{q}{q+1}}) & \text{for } t \ge 2\sqrt{\mu} \end{cases}$$

with ω according to (9.2).

Proof. Since $u_a(\mu) z_a(\mu) = \mu$, we have $u_a(\mu) + z_a(\mu) \ge 2\sqrt{\mu}$ and similarly $u_b(\mu) + z_b(\mu) \ge 2\sqrt{\mu}$. This shows that $\omega_\mu(t) = 0$ for $t < 2\sqrt{\mu}$. For brevity, we set now

$$u_a = u_a(\mu), \quad u_b = u_b(\mu), \quad z_a = z_a(\mu), \quad z_b = z_b(\mu).$$

Now let $t \ge 2\sqrt{\mu}$. Then we have for any $s \ge t$

$$\begin{aligned}
\omega_\mu(t) &\le \text{leb}\left(\{\min(u_a + z_a, u_b + z_b) \le s\}\right) \\
&\le \text{leb}\left(\{\min(\bar{u}_a + \bar{z}_a, \bar{u}_b + \bar{z}_b) - \max(|u_a - \bar{u}_a| + |z_a - \bar{z}_a|, |u_b - \bar{u}_b| + |z_b - \bar{z}_b|) \le s\}\right) \\
&\le \omega(2s) + \text{leb}\left(\{\max(|u_a - \bar{u}_a| + |z_a - \bar{z}_a|, |u_b - \bar{u}_b| + |z_b - \bar{z}_b|) \ge s\}\right) \\
&\le \omega(2s) + \text{leb}\left(\{|u_a - \bar{u}_a| \ge s/2\}\right) + \text{leb}\left(\{|z_a - \bar{z}_a| \ge s/2\}\right) \\
&+ \text{leb}\left(\{|u_b - \bar{u}_b| \ge s/2\}\right) + \text{leb}\left(\{|z_b - \bar{z}_b| \ge s/2\}\right) \\
&\le \omega(2s) + (s/2)^{-q}(\|u_a - \bar{u}_a\|_q^q + \|z_a - \bar{z}_a\|_q^q + \|u_b - \bar{u}_b\|_q^q + \|z_b - \bar{z}_b\|_q^q).
\end{aligned}$$

By Lemma 6.4 there exists a Hölder constant $L > 0$ with $\|w(\mu) - \bar{w}\|_{W_q} \le L\sqrt{\mu}$. This yields

$$\omega_\mu(t) \le \omega(2s) + 4(s/2)^{-q} L^q \mu^{q/2}.$$

Now let $t \ge 2\sqrt{\mu}$. If $t \ge 1$ then we obtain with the choice $s = t$

$$\omega_\mu(t) \le \omega(2t) + 4(t/2)^{-q} L^q \mu^{q/2} \le \omega(2t) + 4L^q \le \omega(2t) + 4L^q t.$$

If $t \le 1$ then the choice $s = t^{q/(q+1)} = t^{1-1/(q+1)}$ yields

$$\omega_\mu(t) \le \omega(2t^{q/(q+1)}) + 4t^{-q+q/(q+1)}(2L)^q \mu^{q/2} \le \omega(2t^{q/(q+1)}) + 4L^q t^{q/(q+1)}.$$

□

We show now that for $\|w-w(\mu)\|_{W_q}$ small enough the result w_+ of the primal-dual Newton step satisfies $\|w_+-w(\mu)\|_{W_p}=o(\|w-w(\mu)\|_{W_q})$, where the estimate is uniform in $\mu\in(0,\mu_0]$. Thus, in contrast to Lemma 8.1 we avoid the μ-dependent convergence factor but obtain a W_p-W_q norm gap. This norm gap will be closed by using a smoothing step.

LEMMA 9.4. *Let $\mu_0>0$ and $\rho_0>0$ be fixed. Assume that (A1)–(A4), (A5)$_q$ hold with some $q\in(p,\infty)$, and that*

$$(v,\hat H(y,u,\lambda)v)\ge\alpha\|v\|_2^2\quad\forall\,v\in L^2(\Omega),\qquad \forall\,w\in N_{-\infty,q}(\mu),\ \|w-w(\mu)\|_{W_q}\le\rho_0,\ \mu\in(0,\mu_0]. \tag{9.6}$$

If in addition $\bar w=\lim_{\mu\searrow 0}w(\mu)$ satisfies strict complementarity then there exists a constant $C>0$ such that for any $0<\mu\le\mu_0$ and for any $w\in N_{-\infty,q}(\mu)$ with $\|w-w(\mu)\|_{W_q}\le\rho_0$ the solution w_+ of the primal dual Newton step (8.1) satisfies

$$\begin{aligned}\|P_\mu(w_+)-w(\mu)\|_{W_p}&\le 2\|w_+-w(\mu)\|_{W_p}\\&\le 2C\left(\omega(4\|w-w(\mu)\|_{W_q}^{\eta q'})^{1/q'}+\|w-w(\mu)\|_{W_q}^{\eta}\right)\|w-w(\mu)\|_{W_q}\\&=o(\|w-w(\mu)\|_{W_q})\end{aligned} \tag{9.7}$$

with

$$\eta=\frac{(q-p)\min(p,q-p)}{pq(p+1)},\quad q'=\frac{qp}{q-p}. \tag{9.8}$$

Moreover, if $\bar w$ satisfies strong strict complementarity (9.4) then

$$\|P_\mu(w_+)-w(\mu)\|_{W_p}\le 2\|w_+-w(\mu)\|_{W_p}\le 2C\|w-w(\mu)\|_{W_q}^{1+\eta} \tag{9.9}$$

with

$$\eta=\frac{\min(1,\kappa)(q-p)\min(p,q-p)}{p^2(q+1)+\min(1,\kappa)p(q-p)}. \tag{9.10}$$

Proof. We know by Corollary 4.3 that $w(\mu)$ is uniformly bounded in W_q for all $\mu\in(0,\mu_0]$. Thus, $\|w-w(\mu)\|_{W_q}\le\rho_0$ implies $\|w\|_{W_q}\le M$ for some constant $M>0$. Hence, all μ,w in (8.2) are contained in a bounded subset of $\{(\mu,w)\in(0,\infty)\times N_{-\infty,q}(\mu)\}$ and Lemma 5.1 yields a constant $C>0$ with

$$\|(S(w)DF_\mu(w))^{-1}\|_{W_t',W_t}\le C,\quad p\le t\le q,$$

where $S(w)$ is the scaling operator (5.1). Furthermore, (8.3) and (8.4) hold. Since the first three components of $F:W_q\to W_p'$ are by (A1), (A5)$_q$ Lipschitz continuously differentiable on bounded subsets, we obtain (8.5), where with a Lipschitz constant $L>0$ the following estimate holds:

$$\|R_1(w(\mu)-w)\|_V+\|R_2(w(\mu)-w)\|_p+\|R_3(w(\mu)-w)\|_\Lambda\le L\|w(\mu)-w\|_{W_q}^2.$$

To obtain an operator with uniformly bounded inverse on the left hand side we multiply with the scaling operator $S(w)$ in (5.1) and obtain

$$S(w)DF_\mu(w)(w_+-w(\mu))=\begin{pmatrix}R_1(w(\mu)-w)\\R_2(w(\mu)-w)\\R_3(w(\mu)-w)\\\frac{(u(\mu)-u)(z_a(\mu)-z_a)}{u_a+z_a}\\\frac{(u(\mu)-u)(z_b(\mu)-z_b)}{u_b+z_b}\end{pmatrix}.$$

Therefore,

$$\|w_+ - w(\mu)\|_{W_p} \le \|(S(w)DF_\mu(w))^{-1}\|_{W_p',W_p}\Bigg(L\|w(\mu)-w\|_{W_q}^2 + \left\|\frac{(u(\mu)-u)(z_a(\mu)-z_a)}{u_a+z_a}\right\|_p + \left\|\frac{(u(\mu)-u)(z_b(\mu)-z_b)}{u_b+z_b}\right\|_p\Bigg).$$

This yields

$$\|w_+ - w(\mu)\|_{W_p} \le C(L\|w-w(\mu)\|_{W_q}^2 + \|R_a\|_p + \|R_b\|_p) \tag{9.11}$$

with

$$R_a := \frac{(u(\mu)-u)(z_a(\mu)-z_a)}{u_a+z_a}, \quad R_b := \frac{(u(\mu)-u)(z_b(\mu)-z_b)}{u_b+z_b}.$$

It remains to estimate $\|R_a\|_p + \|R_b\|_p$. We show first that for $w \in N_{-\infty,q}(\mu)$

$$|R_a| \le \left(1+\frac{1}{2\sqrt{\gamma}}\right)\max(|u-u(\mu)|, |z_a - z_a(\mu)|),$$
$$|R_b| \le \left(1+\frac{1}{2\sqrt{\gamma}}\right)\max(|u-u(\mu)|, |z_b - z_b(\mu)|).$$

To this end we note that $(u(\mu)-a)z_a(\mu) = \mu$, $(u-a)z_a \ge \gamma\mu$. This yields with $u_a, z_a \ge 0$

$$u_a + z_a \ge 2\sqrt{u_a z_a} \ge 2\sqrt{\gamma\mu}.$$

Now we have

$$R_a = \left(\frac{-u_a}{u_a+z_a} + \frac{u_a(\mu)}{u_a+z_a}\right)(z_a(\mu)-z_a) = \left(\frac{-z_a}{u_a+z_a} + \frac{z_a(\mu)}{u_a+z_a}\right)(u_a(\mu)-u_a).$$

Since $u_a(\mu)z_a(\mu) = \mu$ we have $\min(u_a(\mu), z_a(\mu)) \le \sqrt{\mu}$. On $\{u_a(\mu) \le \sqrt{\mu}\}$ we have

$$|R_a| \le \left(1+\frac{1}{2\sqrt{\gamma}}\right)|z_a - z_a(\mu)|$$

and on $\{z_a(\mu) \le \sqrt{\mu}\}$ we have

$$|R_a| \le \left(1+\frac{1}{2\sqrt{\gamma}}\right)|u_a - u_a(\mu)|.$$

The estimate for $|R_b|$ is obtained in the same way.

To estimate $\|R_a\|_p + \|R_b\|_p$ we split Ω for an arbitrary $\beta \in (0, \min(1, (q-p)/p))$ into the sets

$$J = \{u_a + z_a \ge \|w-w(\mu)\|_{W_q}^\beta\}, \quad J^c = \Omega \setminus J.$$

We have with ω_μ in (9.5)

$$\begin{aligned}
\text{leb}\,(\{u_a+z_a \le t\}) &\le \text{leb}\,(\{u_a(\mu)+z_a(\mu) - |u_a-u_a(\mu)| - |z_a - z_a(\mu)| \le t\}) \\
&\le \omega_\mu(2t) + \text{leb}\,(\{|u_a-u_a(\mu)| + |z_a-z_a(\mu)| \ge t\}) \\
&\le \omega_\mu(2t) + \text{leb}\,(\{|u_a-u_a(\mu)| \ge t/2\}) + \text{leb}\,(|z_a-z_a(\mu)| \ge t/2\}) \\
&\le \omega_\mu(2t) + (t/2)^{-q}(\|u_a-u_a(\mu)\|_q^q + \|z_a-z_a(\mu)\|_q^q) \\
&\le \omega_\mu(2t) + 2(t/2)^{-q}\|w-w(\mu)\|_{W_q}^q
\end{aligned}$$

This yields the upper bound for the measure of J^c

$$\text{leb}\,(J^c) \leq \omega_\mu(2\|w-w(\mu)\|_{W_q}^{\beta}) + 2^{q+1}\|w-w(\mu)\|_{W_q}^{(1-\beta)q}.$$

Using $\|uv\|_p \leq \|u\|_{q'}\|v\|_q$, $q' = pq/(q-p)$, we have

$$\|R_a\|_{p,J} \leq \|u-u(\mu)\|_{q',J}\|z_a - z_a(\mu)\|_{q,J}^{1-\beta}.$$

If $q' \leq q$ this yields

$$\|R_a\|_{p,J} \leq c_{q,q'}\|u-u(\mu)\|_{q,J}\|z_a - z_a(\mu)\|_{q,J}^{1-\beta}$$

otherwise $\|u-u(\mu)\|_{q'} \leq \|u-u(\mu)\|_q^{q/q'}\|b-a\|_\infty^{1-q/q'}$ by interpolation and thus

$$\|R_a\|_{p,J} \leq \|b-a\|_\infty^{1-(q-p)/p}\|u-u(\mu)\|_{q,J}^{(q-p)/p}\|z_a - z_a(\mu)\|_{q,J}^{1-\beta}.$$

Combining both cases we obtain

$$\|R_a\|_{p,J} \leq C_1\|w-w(\mu)\|_{W_q}^{1+\min(1,(q-p)/p)-\beta}.$$

On the complement set J^c we obtain by using that $\|1\|_{q',J^c} = \text{leb}\,(J^c)^{1/q'}$

$$\begin{aligned}
\|R_a\|_{p,J^c} &\leq \left(1+\frac{1}{2\sqrt{\gamma}}\right)\|1\|_{q',J^c}\left(\|u-u(\mu)\|_{q,J^c} + \|z_a - z_a(\mu)\|_{q,J^c}\right)\\
&\leq \left(2+\frac{1}{\sqrt{\gamma}}\right)\text{leb}\,(J^c)^{\frac{1}{q'}}\|w-w(\mu)\|_{W_q}\\
&\leq \left(2+\frac{1}{\sqrt{\gamma}}\right)\left(\omega_\mu(2\|w-w(\mu)\|_{W_q}^{\beta}) + 2^{q+1}\|w-w(\mu)\|_{W_q}^{(1-\beta)q}\right)^{\frac{1}{q'}}\|w-w(\mu)\|_{W_q}\\
&\leq \left(2+\frac{1}{\sqrt{\gamma}}\right)\left(\omega_\mu(2\|w-w(\mu)\|_{W_q}^{\beta})^{\frac{1}{q'}} + 2^{\frac{q+1}{q'}}\|w-w(\mu)\|_{W_q}^{\frac{(1-\beta)q}{q'}}\right)\|w-w(\mu)\|_{W_q}.
\end{aligned}$$

Using Lemma 9.3 and $q/q' = (q-p)/p$, we finally obtain constants $C_2, C_3 > 0$ with

$$\begin{aligned}
\|R_a\|_p \leq\; & C_1\|w-w(\mu)\|_{W_q}^{\min(1,(q-p)/p)-\beta}\|w-w(\mu)\|_{W_q}\\
&+ C_2\left(\omega(4\|w-w(\mu)\|_{W_q}^{\beta q/(q+1)})^{(q-p)/(qp)} + \|w-w(\mu)\|_{W_q}^{\beta(q-p)/(p(q+1))}\right)\|w-w(\mu)\|_{W_q}\\
&+ C_3\|w-w(\mu)\|_{W_q}^{(1-\beta)(q-p)/p}\|w-w(\mu)\|_{W_q}.
\end{aligned}$$

The last term has at least the order of the first term. Balancing the orders of the first and second term leads to

$$\beta = \frac{(q+1)\min(p, q-p)}{q(p+1)}$$

and results in the estimate

$$\|R_a\|_p \leq C'\left(\omega(4\|w-w(\mu)\|_{W_q}^{\eta q'})^{1/q'} + \|w-w(\mu)\|_{W_q}^{\eta}\right)\|w-w(\mu)\|_{W_q}$$

with

$$\eta = \frac{(q-p)\min(p,q-p)}{pq(p+1)}, \quad q' = \frac{qp}{q-p}.$$

The same estimate is valid for $\|R_b\|_p$. Inserting this in (9.11) yields (9.7).

If in addition strong strict complementarity (9.4) holds, i.e., $\omega(t) \le C_c t^\kappa$, then the middle term has order $O(\|w - w(\mu)\|_{W_q}^{\min(1,\kappa)\beta(q-p)/(p(q+1))})$ and balancing with the first term gives

$$\beta = \frac{(q+1)\min(p, q-p)}{p(q+1) + \min(1,\kappa)(q-p)}.$$

Inserting this choice of β leads to the asserted estimate (9.9). □

We see that a norm gap occurs in the estimates (9.7), (9.9). To close the norm gap we will use a smoothing step.

9.2. Smoothing steps. We construct now an operator

$$Q_\mu : (y, u, \lambda, z_a, z_b) \in W_p \mapsto (y, \tilde{u}, \lambda, \tilde{z}_a, \tilde{z}_b) \in W_q$$

such that there exists a constant $L_S > 0$ with

$$\|Q_\mu(w) - w(\mu)\|_{W_q} \le L_S \|w - w(\mu)\|_{W_p} \tag{9.12}$$

to close the norm gap in (9.7). Let (A1)-(A5)$_q$ and the assumptions of Lemma 6.4 hold. Then $w(\mu)$ is the unique solution of (2.9) and satisfies with the notations of (A5)$_q$, 5. in particular

$$\begin{aligned} 0 = \ell_u(w(\mu)) &= \beta(u(\mu)) + \hat{g}_s(y(\mu), u(\mu), \lambda(\mu)) + z_b(\mu) - z_a(\mu) \\ &= \beta(u(\mu)) + \frac{\mu}{b - u(\mu)} - \frac{\mu}{u(\mu) - a} + \hat{g}_s(y(\mu), u(\mu), \lambda(\mu)) \\ &=: \beta_\mu(u(\mu)) + \hat{g}_s(y(\mu), u(\mu), \lambda(\mu)), \end{aligned}$$

where $\beta \in C^1(\mathbb{R}),\ \beta' \ge \alpha_0 > 0$ and where

$$\hat{g}_s : (y, u, \lambda) \in Y \times U \times \Sigma \mapsto J_u(y, u) - \beta(u) + c_u(y, u)^* \lambda \in L^q(\Omega)$$

is by (A5)$_q$, 5. well defined and Lipschitz on bounded sets. Since the mappings

$$\beta_{\mu;\xi} : t \in (a(\xi), b(\xi)) \mapsto \beta(t) + \frac{\mu}{b(\xi) - t} - \frac{\mu}{t - a(\xi)}$$

satisfy $\beta'_{\mu;\xi} \ge \alpha_0 > 0$ and $\beta_{\mu;\xi}((a(\xi), b(\xi))) = \mathbb{R}$, the inverse mappings $\beta_{\mu;\xi}^{-1} : \mathbb{R} \to (a(\xi), b(\xi))$ exist and are Lipschitz continuous with Lipschitz constant $\le 1/\alpha_0$. Thus, also the mapping

$$\beta_\mu : u \in \{v \ :\ \ a < v < b\} \mapsto \beta_\mu(u) = \beta(u) + \frac{\mu}{b - u} - \frac{\mu}{u - a}$$

has a Lipschitz continuous inverse

$$\beta_\mu^{-1} : L^q(\Omega) \to (\{v \ :\ \ a < v < b\}, \|\cdot\|_q)$$

and we have

$$u(\mu) = \beta_\mu^{-1}(-\hat{g}_s(y(\mu), u(\mu), \lambda(\mu))).$$

Thus, given any $w \in W_q$ the "smoothed" control

$$u^s := \beta_\mu^{-1}(-\hat{g}_s(y, u, \lambda)) \tag{9.13}$$

satisfies

$$\|u^s - u(\mu)\|_q \le \frac{1}{\alpha_0} \|\hat{g}_s(y, u, \lambda) - \hat{g}_s(y(\mu), u(\mu), \lambda(\mu))\|_q \le \frac{L_g}{\alpha_0} \|w - w(\mu)\|_{W_p} \tag{9.14}$$

with a Lipschitz constant L_g of $\hat{g}_s$ according to (A5)$_q$, 5.

Smoothing of the z_a, z_b-components can now be obtained by using the identities

$$z_a(\mu) = \frac{\mu}{u(\mu) - a}, \quad z_b(\mu) = \frac{\mu}{b - u(\mu)}. \tag{9.15}$$

In fact, after computing u^s we choose z_a^s, z_b^s according to

$$z_a^s = \frac{\mu}{u^s - a}, \qquad z_b^s = \frac{\mu}{b - u^s}. \tag{9.16}$$

We will see that this leads in fact to an operator that has the smoothing property (9.12).

DEFINITION 9.5. *The smoothing operator $Q_\mu : W_p \to W_q$ is defined by*

$$Q_\mu(y, u, \lambda, z_a, z_b) = (y, u^s, \lambda, z_a^s, z_b^s)$$

with u^s according to (9.13) and z_a^s, z_b^s according to (9.16), where $\hat{g}_s(y, u, \lambda) = J_u(y, u) - \beta(u) + c_u(y, u)^* \lambda$ according to (A5)$_q$, 5.

THEOREM 9.6. *Let the assumptions of Lemma 6.3 hold. Then for any $\mu_0 > 0$, $\rho_0 > 0$ there is a constant $L_S > 0$ such that the smoothing operator $Q_\mu : W_p \to W_q$ of Definition 9.5 is well defined and satisfies*

$$\|Q_\mu(w) - w(\mu)\|_{W_q} \le L_S \|w - w(\mu)\|_{W_p} \quad \forall\, w \in W_p,\; \|w - w(\mu)\|_{W_p} \le \rho_0, \quad \forall\, \mu \in (0, \mu_0]. \tag{9.17}$$

Moreover, $(y, u^s, \lambda, z_a^s, z_b^s) = Q_\mu(w)$ satisfies

$$a < u^s < b, \quad (u^s - a) z_a^s = \mu, \quad (b - u^s) z_b^s = \mu$$

and thus $Q_\mu(w) \in N_{-\infty,q}(\mu)$.

Proof. We have already shown that (9.14) holds, where L_g is the Lipschitz constant of $\hat{g}_s$ on the bounded set of all $w \in W_p$ considered in (9.17). Moreover, we have by (9.16) and the choice (9.13) of u^s

$$z_b^s - z_a^s = -\beta(u^s) - \hat{g}_s(y, u, \lambda). \tag{9.18}$$

On the other hand

$$z_b(\mu) - z_a(\mu) = -\beta(u(\mu)) - \hat{g}_s(y(\mu), u(\mu), \lambda(\mu)). \tag{9.19}$$

Now consider the following partition of Ω

$$\Omega_a = \{u^s \ge (b+a)/2,\; u(\mu) \ge (b+a)/2\}, \quad \Omega_b = \{u^s < (b+a)/2,\; u(\mu) < (b+a)/2\},$$
$$\Omega_a' = \{u^s \ge (b+a)/2,\; u(\mu) < (b+a)/2\}, \quad \Omega_b' = \{u^s < (b+a)/2,\; u(\mu) \ge (b+a)/2\}.$$

Then we obtain on Ω_a by (9.16) and (9.19)

$$|(z_a^s - z_a(\mu))|_{\Omega_a}| = \left| \frac{\mu}{u^s - a} - \frac{\mu}{u(\mu) - a} \right|_{\Omega_a} = \left. \frac{\mu |u^s - u(\mu)|}{(u^s - a)(u(\mu) - a)} \right|_{\Omega_a} \le \frac{4\mu}{\nu^2} |(u^s - u(\mu))_{\Omega_a}|$$

Now (9.18), (9.19) yield

$$\begin{aligned} |(z_b^s - z_b(\mu))_{\Omega_a}| &\le |(z_a^s - z_a(\mu))_{\Omega_a}| + |(\beta(u^s) - \beta(u(\mu)))_{\Omega_a}| \\ &\quad + |(\hat{g}_s(y, u, \lambda) - \hat{g}_s(y(\mu), u(\mu), \lambda(\mu)))_{\Omega_a}|. \end{aligned}$$

Completely similar we obtain on Ω_b by (9.16) and (9.19)

$$|(z_b^s - z_b(\mu))_{\Omega_b}| = \left|\frac{\mu}{b-u^s} - \frac{\mu}{b-u(\mu)}\right|_{\Omega_b} \le \frac{4\mu}{\nu^2}|(u^s - u(\mu))_{\Omega_b}|$$

Now again (9.18), (9.19) yield

$$\begin{aligned}|(z_a^s - z_a(\mu))_{\Omega_b}| &\le |(z_b^s - z_b(\mu))_{\Omega_b}| + |(\beta(u^s) - \beta(u(\mu)))_{\Omega_b}| \\ &\quad + |(\hat{g}_s(y,u,\lambda) - \hat{g}_s(y(\mu),u(\mu),\lambda(\mu)))_{\Omega_b}|.\end{aligned}$$

Finally, (9.16), (9.15) yield on Ω_a'

$$(z_a^s - z_a(\mu))_{\Omega_a'} < 0, \quad (z_b^s - z_b(\mu))_{\Omega_a'} > 0$$

and thus the difference of (9.18), (9.19) yields

$$\begin{aligned}&|(z_b^s - z_b(\mu))_{\Omega_a'}| + |(z_a^s - z_a(\mu))_{\Omega_a'}| = |(z_b^s - z_b(\mu) + z_a(\mu) - z_a^s)_{\Omega_a'}| \\ &\le |(\beta(u^s) - \beta(u(\mu)))_{\Omega_a'}| + |(\hat{g}_s(y,u,\lambda) - \hat{g}_s(y(\mu),u(\mu),\lambda(\mu)))_{\Omega_a'}|.\end{aligned}$$

Analogously we obtain on Ω_b'

$$\begin{aligned}&|(z_b^s - z_b(\mu))_{\Omega_b'}| + |(z_a^s - z_a(\mu))_{\Omega_b'}| = |(z_b^s - z_b(\mu) + z_a(\mu) - z_a^s)_{\Omega_b'}| \\ &\le |(\beta(u^s) - \beta(u(\mu)))_{\Omega_b'}| + |(\hat{g}_s(y,u,\lambda) - \hat{g}_s(y(\mu),u(\mu),\lambda(\mu)))_{\Omega_b'}|.\end{aligned}$$

Taking all together, we have shown that

$$\begin{aligned}&|z_b^s - z_b(\mu)| + |z_a^s - z_a(\mu)| \\ &\le |\beta(u^s) - \beta(u(\mu))| + |\hat{g}_s(y,u,\lambda) - \hat{g}_s(y(\mu),u(\mu),\lambda(\mu))| + \frac{8\mu_0}{\nu^2}|u^s - u(\mu)| \\ &\le |\hat{g}_s(y,u,\lambda) - \hat{g}_s(y(\mu),u(\mu),\lambda(\mu))| + \left(\frac{8\mu_0}{\nu^2} + \sup_{[\inf a, \sup b]} \beta'\right)|u^s - u(\mu)|.\end{aligned}$$

Hence, (9.14) yields

$$\|z_b^s - z_b(\mu)\|_q + \|z_a^s - z_a(\mu)\|_q \le \left(L_g + \frac{L_g}{\alpha_0}\left(\frac{8\mu_0}{\nu^2} + \sup_{[\inf a, \sup b]} \beta'\right)\right)\|w - w(\mu)\|_{W_p}.$$

Together with (9.14), (9.17) is proven. The last statement is obvious by the definition of Q_μ. ☐

If we replace the projection P_μ in Lemma 9.4 by the smoothing operator Q_μ (which yields a point in the neighborhood) then we obtain directly the following corollary.

COROLLARY 9.7. *Let the assumptions of Lemma 9.4 hold. If $\bar{w} = \lim_{\mu \searrow 0} w(\mu)$ satisfies strict complementarity then there exist constants $L_S, C > 0$ such that for any $0 < \mu \le \mu_0$ and for any $w \in N_{-\infty,q}(\mu)$ with $\|w - w(\mu)\|_{W_q} \le \rho_0$ the solution w_+ of the primal dual Newton step (8.1) satisfies*

(9.20)

$$\begin{aligned}\|Q_\mu(w_+) - w(\mu)\|_{W_q} &\le L_S\|w_+ - w(\mu)\|_{W_p} \\ &\le 2L_S C\left(\omega(4\|w - w(\mu)\|_{W_q}^{\eta q'})^{1/q'} + \|w - w(\mu)\|_{W_q}^{\eta}\right)\|w - w(\mu)\|_{W_q} \\ &= o(\|w - w(\mu)\|_{W_q})\end{aligned}$$

with

$$\eta = \frac{(q-p)\min(p,q-p)}{pq(p+1)}, \quad q' = \frac{qp}{q-p}.$$

Moreover, if $\bar{w}$ satisfies strong strict complementarity (9.4) then

$$\|Q_\mu(w_+) - w(\mu)\|_{W_q} \le L_S \|w_+ - w(\mu)\|_{W_p} \le 2 L_S C \|w - w(\mu)\|_{W_q}^{1+\eta} \tag{9.21}$$

with

$$\eta = \frac{\min(1,\kappa)(q-p)\min(p,q-p)}{p^2(q+1) + \min(1,\kappa)p(q-p)}.$$

Note that no norm gap appears in (9.20) and (9.21).

9.3. A modified interior-point method with smoothing step. We consider now the following modification of Algorithm PDPF.

Algorithm PDPFS: Primal-Dual Interior-Point Method with Smoothing.

1. Choose $\mu_0 > 0$, $\gamma \in (0,1)$, $\mu_{-\infty} > 0$, $(\sigma_{\text{min},k}) \subset (0,1)$, $\mu_0 \in (0, \mu_{-\infty}]$, and

$$w_0 := (y_0, u_0, \lambda_0, z_{a,0}, z_{b,0}) \in N_{-\infty,q}(\mu_0).$$

   Set $k := 0$.
2. Solve the primal-dual Newton system

$$DF_{\mu_k}(w_k) s_k = -F_{\mu_k}(w_k)$$

   and set $w_{k+1} := Q_{\mu_k}(w_k + s_k)$.
3. Choose $\sigma_k \in [\sigma_{\text{min},k}, 1]$ and set $\mu_{k+1} := \sigma_k \mu_k$.
4. Set $k := k + 1$ and goto step 2.

REMARK 9.8. *As already algorithm PDPF, we have stated this algorithm in a very basic form.* We now will prove global linear and local superlinear convergence of algorithm PDPFS.

THEOREM 9.9. *Let $\mu_0 > 0$ and $\rho_0 > 0$ be fixed. Assume that (A1)–(A4) and (A5)$_q$ with some $q \in (p, \infty)$ hold, that $u \in \mathcal{B} \mapsto J(y(u), u)$ is convex (to ensure uniqueness of the central path), and that $\bar{w} = \lim_{\mu \searrow 0} w(\mu)$ satisfies strict complementarity. Furthermore, assume that*

$$\begin{aligned} (v, \hat{H}(y,u,\lambda)v) \ge \alpha \|v\|_2^2 \quad & \forall\, v \in L^2(\Omega), \\ & \forall\, w \in N_{-\infty,q}(\mu),\ \|w - w(\mu)\|_{W_q} \le \rho_0,\ \mu \in (0, \mu_0]. \end{aligned} \tag{9.6}$$

Then there exist constants $\bar{\rho} \in (0, \rho_0]$, $\bar{\sigma}_{\text{max}} \in (0,1)$ and a sequence $\sigma_{\text{max}} > \bar{\sigma}_{\text{min},k} \searrow 0$ such that Algorithm PFPFS has the following convergence property:

For all initial points $w_0 \in N_{-\infty,q}(\mu_0)$ with $\|w_0 - w(\mu_0)\|_{W_q} \le \bar{\rho}$ and every choice $\sigma_{\text{min},k} \in [\bar{\sigma}_{\text{min},k}, \sigma_{\text{max}}]$, Algorithm PDPFS generates a sequence with

$$\|w_k - w(\mu_k)\|_{W_q} \le C\sqrt{\mu_{k-1}} \tag{9.22}$$

$$\|w_k - \bar{w}\|_{W_q} \le (C + L)\sqrt{\mu_{k-1}} \tag{9.23}$$

$$\mu_k = \sigma_0 \cdots \sigma_{k-1} \mu_0 \le \sigma_{\text{max}}^k \mu_0 \to 0. \tag{9.24}$$

Here, $C > 0$ is a constant and L is the Hölder constant of the central path on $(0, \mu_0]$. This shows global convergence with R-linear rate.

The above conditions allow to choose $0 < \sigma_{\min,k} \le \sigma_k \searrow 0$, which yields R-superlinear convergence.

If strong strict complementarity (9.4) holds at $\bar{w}$, then

$$\bar{\sigma}_{\min,k} = O(\mu_k^\eta)$$

with $\eta > 0$ according to (9.10) and the choice $\sigma_{\min,k} = O(\bar{\sigma}_{\min,k})$ yields R-superlinear convergence with rate $1+\eta$.

Proof. Consider an arbitrary $\mu \in (0, \mu_0]$. Then there exists by Corollary 9.7 a constant $C > 0$ such that for any $w \in N_{-\infty,q}(\mu)$ with $\|w - w(\mu)\|_{W_q} \le \rho_0$ the estimate holds

$$\|Q_\mu(w_+) - w(\mu)\|_{W_q} \le \psi(\|w_+ - w(\mu)\|_{W_q})\|w_+ - w(\mu)\|_{W_q} \tag{9.25}$$

with

$$\psi(t) = 2L_S C(\omega(4t^{\eta q'})^{1/q'} + t^\eta) \tag{9.26}$$

and $\eta > 0$, q' according to (9.8), where w_+ is the result of the primal-dual Newton step (8.1).

Choose $\tau > 0$ such that

$$\psi(\tau) < \frac{1}{2}$$

and set

$$\bar{\rho} = \min(\tau, 2L\sqrt{\mu_0}) \tag{9.27}$$

with the Hölder constant L of the central path. Now let Algorithm PDPFS use the iterates

$$w_{k+1} = Q_{\mu_k}(w_k + s_k), \quad \mu_{k+1} = \sigma_k \mu_k, \tag{9.28}$$

where σ_k satisfies

$$\sigma_k \ge \bar{\sigma}_{\min,k} := \begin{cases} \max\left\{\left(1 - \frac{\tau(1-\psi(\tau))}{L\sqrt{\mu_k}}\right)_+^2, 4\psi(\tau)^2\right\} & \text{if } \tau < 2L\sqrt{\mu_k}, \\ 4\psi(2L\sqrt{\mu_k})^2 & \text{if } \tau \ge 2L\sqrt{\mu_k}. \end{cases} \tag{9.29}$$

We note that $0 < \bar{\sigma}_{\min,k} < 1$, more precisely,

$$\bar{\sigma}_{\min,k} \le \max\left\{\left(1 - \frac{\tau(1-\psi(\tau))}{L\sqrt{\mu_0}}\right)_+^2, 4\psi(\tau)^2\right\} =: \bar{\sigma}_{\min} < 1.$$

We show next that the choice (9.29) yields

$$\|w_k - w(\mu_k)\|_{W_q} \le \min(\tau, 2L\sqrt{\mu_{k-1}}), \tag{9.30}$$

$$\|w_k - w(\mu_{k-1})\|_{W_q} \le \min(\tau, 2L\sqrt{\mu_{k-1}}), \tag{9.31}$$

where we set $\mu_{-1} = \mu_0$. In fact, this is true for $k = 0$ if $\|w_0 - w(\mu_0)\|_{W_q} \le \bar{\rho}$. Moreover, to proceed by induction we observe that

$$\begin{aligned} \|w_{k+1} - w(\mu_{k+1})\|_{W_q} &\le \|w(\mu_{k+1}) - w(\mu_k)\|_{W_q} + \|w_{k+1} - w(\mu_k)\|_{W_q} \\ &\le L(\sqrt{\mu_k} - \sqrt{\mu_{k+1}}) + \psi(\|w_k - w(\mu_k)\|_{W_q})\|w_k - w(\mu_k)\|_{W_q} \\ &\le L(1 - \sqrt{\sigma_k})\sqrt{\mu_k} + \psi(\min(\tau, 2L\sqrt{\mu_{k-1}}))\min(\tau, 2L\sqrt{\mu_{k-1}}). \end{aligned}$$

We consider now three cases.

Case 1: $\tau < 2L\sqrt{\mu_k}$. Then we obtain by using (9.29)

$$\begin{aligned}\|w_{k+1} - w(\mu_{k+1})\|_{W_q} &\le \|w(\mu_{k+1}) - w(\mu_k)\|_{W_q} + \|w_{k+1} - w(\mu_k)\|_{W_q} \\ &\le L(1 - \sqrt{\sigma_k})\sqrt{\mu_k} + \psi(\tau)\tau \le \tau = \min(\tau, 2L\sqrt{\mu_k}).\end{aligned}$$

Case 2: $2L\sqrt{\mu_k} \le \tau < 2L\sqrt{\mu_{k-1}}$. Then (9.29) yields $\mu_k \ge 4\psi(\tau)^2\mu_{k-1}$ and thus

$$\begin{aligned}\|w_{k+1} - w(\mu_{k+1})\|_{W_q} &\le \|w(\mu_{k+1}) - w(\mu_k)\|_{W_q} + \|w_{k+1} - w(\mu_k)\|_{W_q} \\ &\le L(1 - \sqrt{\sigma_k})\sqrt{\mu_k} + \psi(\tau)\tau \le L\sqrt{\mu_k} + \psi(\tau)2L\sqrt{\mu_{k-1}} \\ &\le 2L\sqrt{\mu_k} = \min(\tau, 2L\sqrt{\mu_k}).\end{aligned}$$

Case 3: $2L\sqrt{\mu_{k-1}} \le \tau$. Then (9.29) yields $\mu_k \ge 4\psi(2L\sqrt{\mu_{k-1}})^2\mu_{k-1}$ and thus

$$\begin{aligned}\|w_{k+1} - w(\mu_{k+1})\|_{W_q} &\le \|w(\mu_{k+1}) - w(\mu_k)\|_{W_q} + \|w_{k+1} - w(\mu_k)\|_{W_q} \\ &\le L(1 - \sqrt{\sigma_k})\sqrt{\mu_k} + \psi(2L\sqrt{\mu_{k-1}})2L\sqrt{\mu_{k-1}} \\ &\le L\sqrt{\mu_k} + L\sqrt{\mu_k} \le 2L\sqrt{\mu_k} = \min(\tau, 2L\sqrt{\mu_k}).\end{aligned}$$

Hence, in all three cases (9.30), (9.31) hold. This shows (9.22) and with the Hölder constant L of the central path also (9.23), since

$$\|w_k - \bar{w}\|_{W_q} \le \|w_k - w(\mu_{k-1})\|_{W_q} + \|w(\mu_{k-1}) - \bar{w}\|_{W_q} \le \min(\tau, 2L\sqrt{\mu_{k-1}}) + L\sqrt{\mu_{k-1}}.$$

(9.24) is obvious from $\mu_{k+1} = \sigma_k\mu_k$ and yields R-linear convergence.

Moreover, (9.29) shows now that for k large enough $\bar{\sigma}_{\min,k} = 4\psi(2L\sqrt{\mu_k})^2 \to 0$ and therefore for any constant $\kappa > 1$ the choice $\sigma_k \in [\bar{\sigma}_{\min,k}, \min(\kappa\bar{\sigma}_{\min,k}, \sigma_{\max})]$ yields $\mu_{k+1} = o(\mu_k)$ and thus R-superlinear convergence.

Finally, let strong strict complementarity be satisfied. Then there exists by Corollary 9.7 a constant $C > 0$ such that (9.25) is satisfied with with

$$\psi(t) = 2L_S C t^\eta \tag{9.32}$$

and $\eta > 0$ according to (9.10). Now (9.30), (9.31) can be shown exactly as above for the definition (9.32) of ψ and the corresponding $\bar{\sigma}_{\min,k}$ according to (9.29). Moreover, (9.29) shows now that for k large enough $\bar{\sigma}_{\min,k} = 4\psi(2L\sqrt{\mu_k})^2 = O(\mu_k^\eta) \to 0$ and therefore for any constant $\kappa > 1$ the choice $\sigma_k \in [\bar{\sigma}_{\min,k}, \min(\kappa\bar{\sigma}_{\min,k}, \sigma_{\max})]$ yields $\mu_{k+1} = O(\mu_k^{1+\eta})$. □

10. Numerical results. We conclude the paper by a demonstration of the practical performance of the proposed method. We consider the elliptic control problem (2.2) with

$$\begin{aligned}\Omega = (0,1)^2, &\text{ i.e., } n = 2, \\ a = -5, \quad b = 0, &\quad \alpha = 0.001, \\ y_d = \sin(\pi\xi_1)&\sin(\pi\xi_2)\cos(3\xi_1).\end{aligned}$$

Both, the first and the second setting are applicable. In the first setting ($Y = H_0^1(\Omega)$, etc.), our assumptions are satisfied for any value $q \in (p, \infty)$. In the second setting ($Y = H_0^1(\Omega) \cap H^2(\Omega)$, etc.), our assumptions are satisfied for any value $q \in (p, \infty]$.

In our implementation of algorithm PDPFS, the adjustment of μ_k, and thus the choice of σ_k, is done adaptively based on the value of μ_k. It turns out to be beneficial (sometimes one iteration is

It.	$\|F_{\mu_k}(w^k)\|_{W'_\infty}$	μ_k
Grid 4: 30529 Nodes		
0	8.60e+01	2.50e+00
1	3.23e-07	2.50e-01
2	7.86e-02	1.57e-02
3	7.75e-01	3.95e-04
4	7.93e-01	2.90e-06
5	1.10e-01	4.13e-09
6	1.66e-03	6.62e-13
7	5.05e-09	

TABLE 10.1
Results for algorithm PDPFS applied on the finest grid.

saved) to compute not only the smoothing step, but also the projected step, and to use the latter if it reduces the norm of F_{μ_k} significantly more than the smoothing step.

The discretization is done by linear finite elements. We generate a hierarchy of 5 triangular grids, the coarsest with 139 nodes, the finest with 30529 nodes. Starting from the coarsest, the next finer grid is obtained by uniform refinement of the previous grid. We perform two numerical tests, the second (nested iteration) being of higher practical relevance.

In the first test we solve the fine grid problem directly with the PDPFS method from the following initial point:

$$y_h^0 = y_{d,h}, \quad \lambda_h^0 = 0, \quad (u_h^0, z_{a,h}^0, z_{b,h}^0) \text{ obtained by one smoothing step.}$$

Here, the subscript h indicates that we are working in finite element space, and $y_{d,h}$ is the nodewise interpolation of y_d.

In the second test, we use nested iteration, i.e., we start on the coarsest grid, solve the discretized problem approximately, and prolongate the solution to obtain an initial point on the next finer level. This is repeated until the problem on the finest grid is solved up to the desired accuracy. The accuracy requirements are increased from grid to grid. The prolongation is performed for the state and the adjoint state only, the initial control u_h^0 and the initial multipliers $z_{a,h}^0$, $z_{b,h}^0$ are obtained by applying a smoothing step. The approximate solution of the discrete problems on the various grids is done by a discrete version of Algorithm PDPFS. The inital μ-value is computed from u_h^0, $z_{a,h}^0$, and $z_{b,h}^0$.

The results of the first test can be found in Table 10.1. We observe fast local convergence of μ_k and of the residual as predicted by the theory.

The results for the second test in Table 10.2 show that nested iteration is very efficient. Thanks to the smoothing step, we obtain very good initial points, since only the smooth parts y_h and λ_h of the coarse grid solution are prolongated. On the finer grids, we need only two iterations, which is a very good performance.

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It.	$\|F_{\mu_k}(w^k)\|_{W'_\infty}$	μ_k
Grid 0: 139 Nodes		
0	1.89e+01	2.50e+00
1	2.47e-07	2.50e-01
2	7.31e-02	1.57e-02
3	7.37e-01	3.95e-04
4	6.08e-01	2.89e-06
5	7.43e-02	4.13e-09
6	1.38e-03	
Grid 1: 513 Nodes		
0	1.34e+00	7.22e-06
1	1.15e-03	1.40e-08
2	1.93e-03	3.36e-12
3	2.61e-06	

It.	$\|F_{\mu_k}(w^k)\|_{W'_\infty}$	μ_k
Grid 2: 1969 Nodes		
0	1.62e+00	1.09e-06
1	9.19e-05	1.12e-09
2	3.16e-07	
Grid 3: 7713 Nodes		
0	1.75e+00	1.67e-07
1	1.59e-05	9.18e-11
2	4.42e-08	
Grid 4: 30529 Nodes		
0	1.80e+00	1.93e-08
1	1.49e-06	5.17e-12
2	4.86e-09	

TABLE 10.2
Results for the nested iteration.

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